Earth's Free Oscillations

or

Learning to love normal mode seismology

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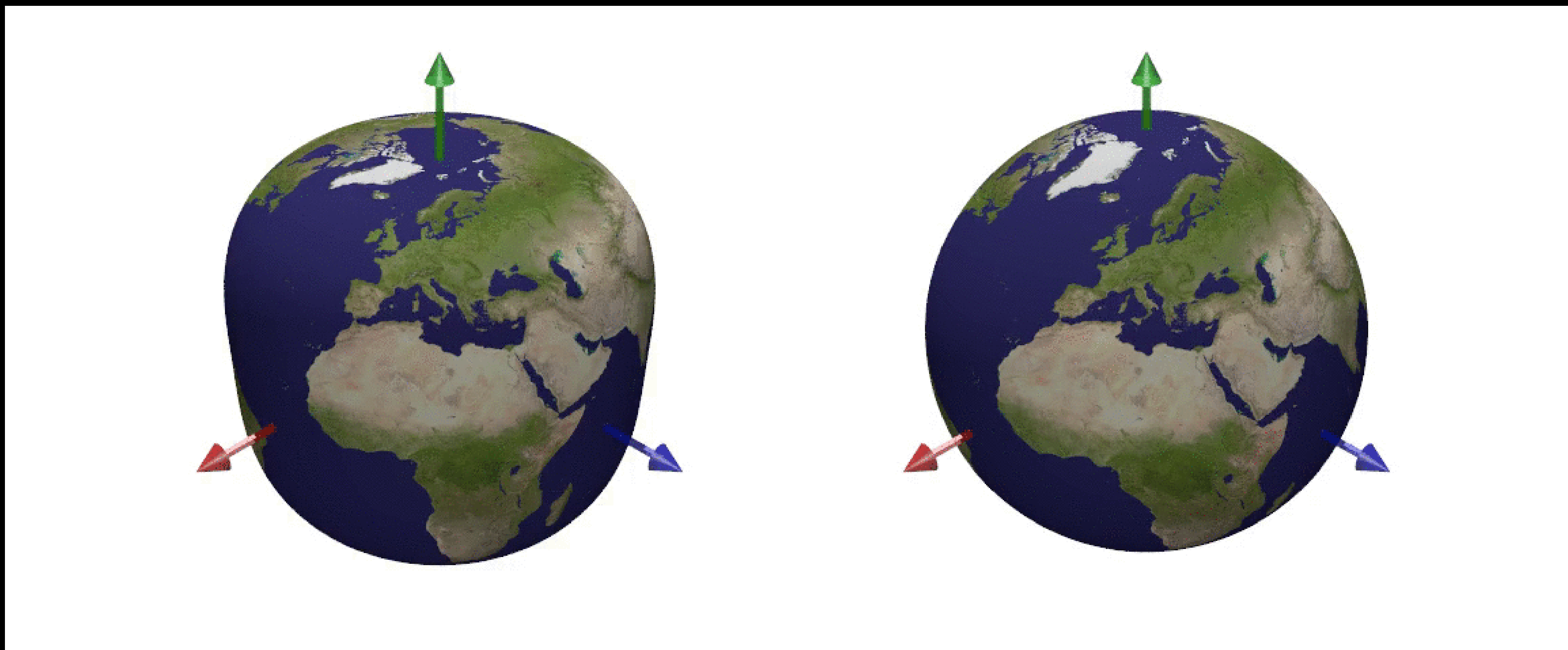
Normal modes



Spheroidal, n=0, ℓ=4

period ≈ 26 minutes

Toroidal, n=0, ℓ=4

period ≈ 22 minutes

Animations from http://lucien.saviot.free.fr/terre/index.en.html

A normal mode spectrum

We look at normal modes in the frequency domain. Here is an observation of some of the lower frequency normal mode oscillations generated by the 2004 Sumatra Earthquake.

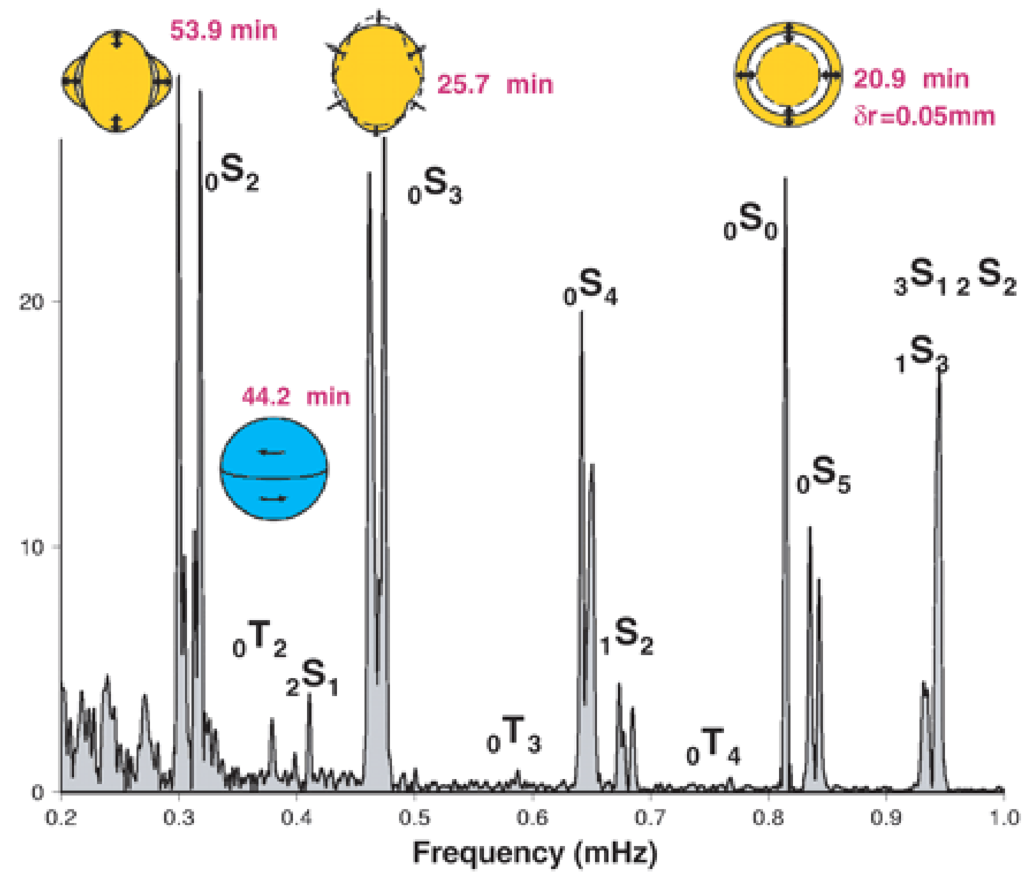


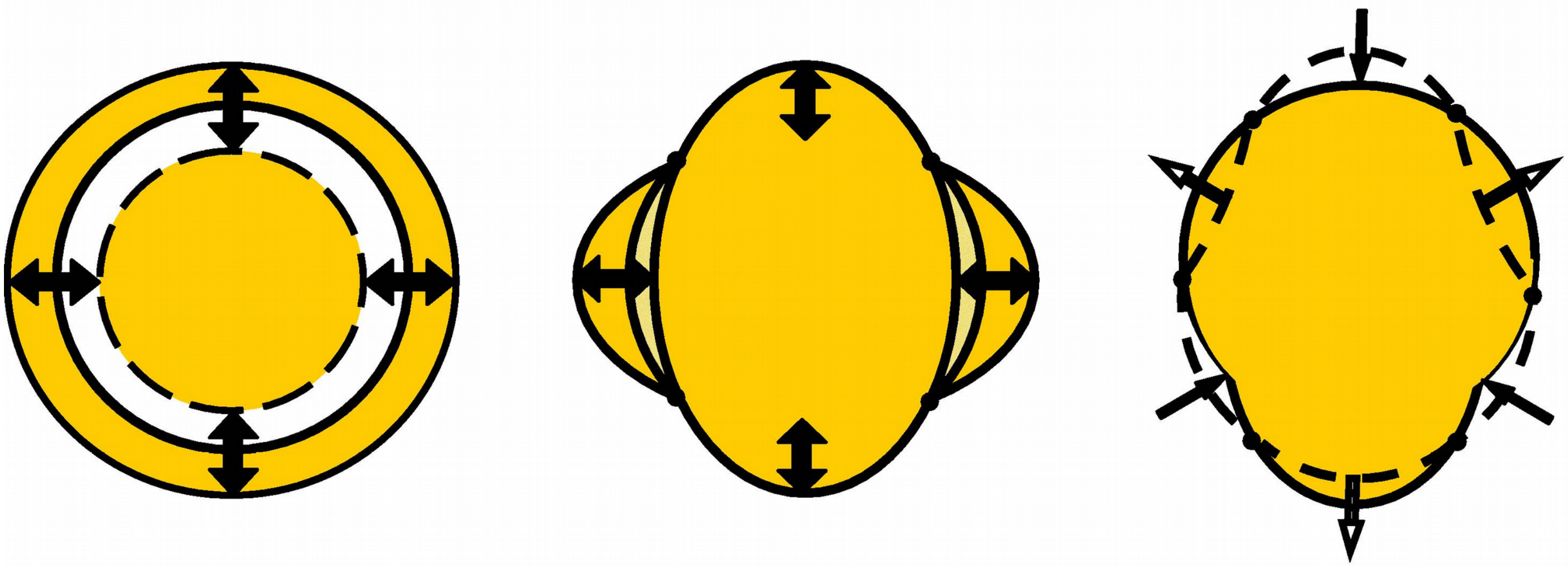
Figure from Park et al, Science 20 May 2005: 308 (5725), 1139-1144

Splitting of the peaks, so they are not at one distinct frequency, is caused by Earth's rotation, ellipticity, anisotropy and lateral heterogeneity – see second half of the lecture.

Spheroidal normal modes

There are two types of normal modes.

- Spheroidal modes are analogous to modes with P-SV motion.

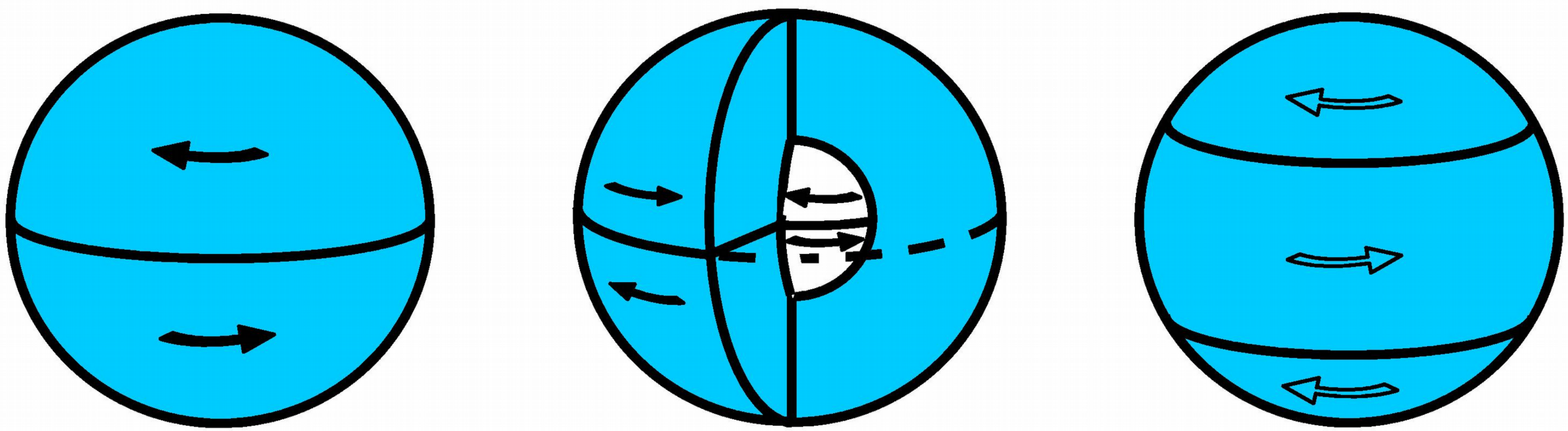


Spheroidal modes ${}_0S_0$ (20.5 min), ${}_0S_2$ (53.9 min) and ${}_0S_3$ (25.7 min)

Figure from http://www.iris.iris.edu/sumatra/free_oscillations_second.htm

Toroidal normal modes

Toroidal modes are analagous to Love waves or SH motion.
They are labelled by

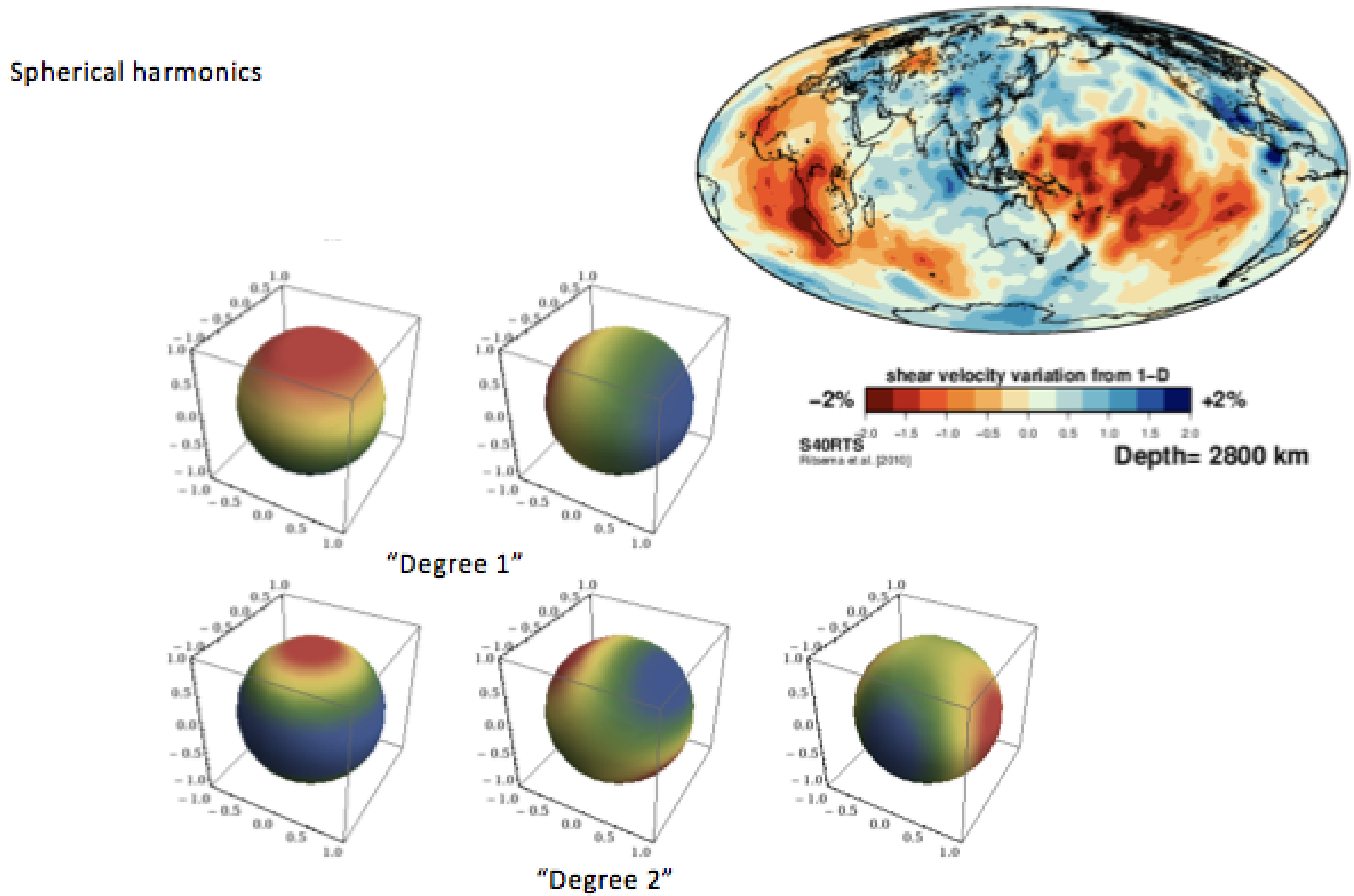


Toroidal modes ${}_0T_2$ (44.2 min), ${}_1T_2$ (12.6 min) and ${}_0T_3$ (28.4 min)

The normal modes of a string - a "1D" object - with fixed endpoints look like sines and cosines.

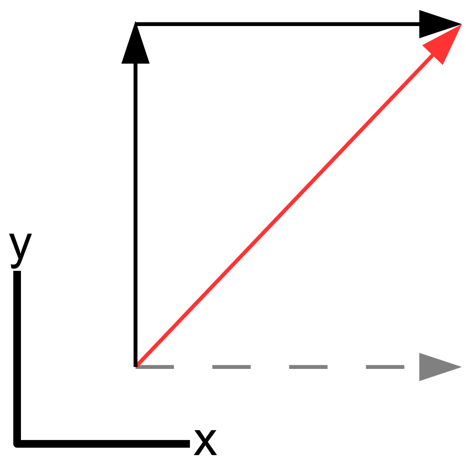
Figure from http://www.iris.iris.edu/sumatra/free_oscillations_second.htm

Spherical harmonics



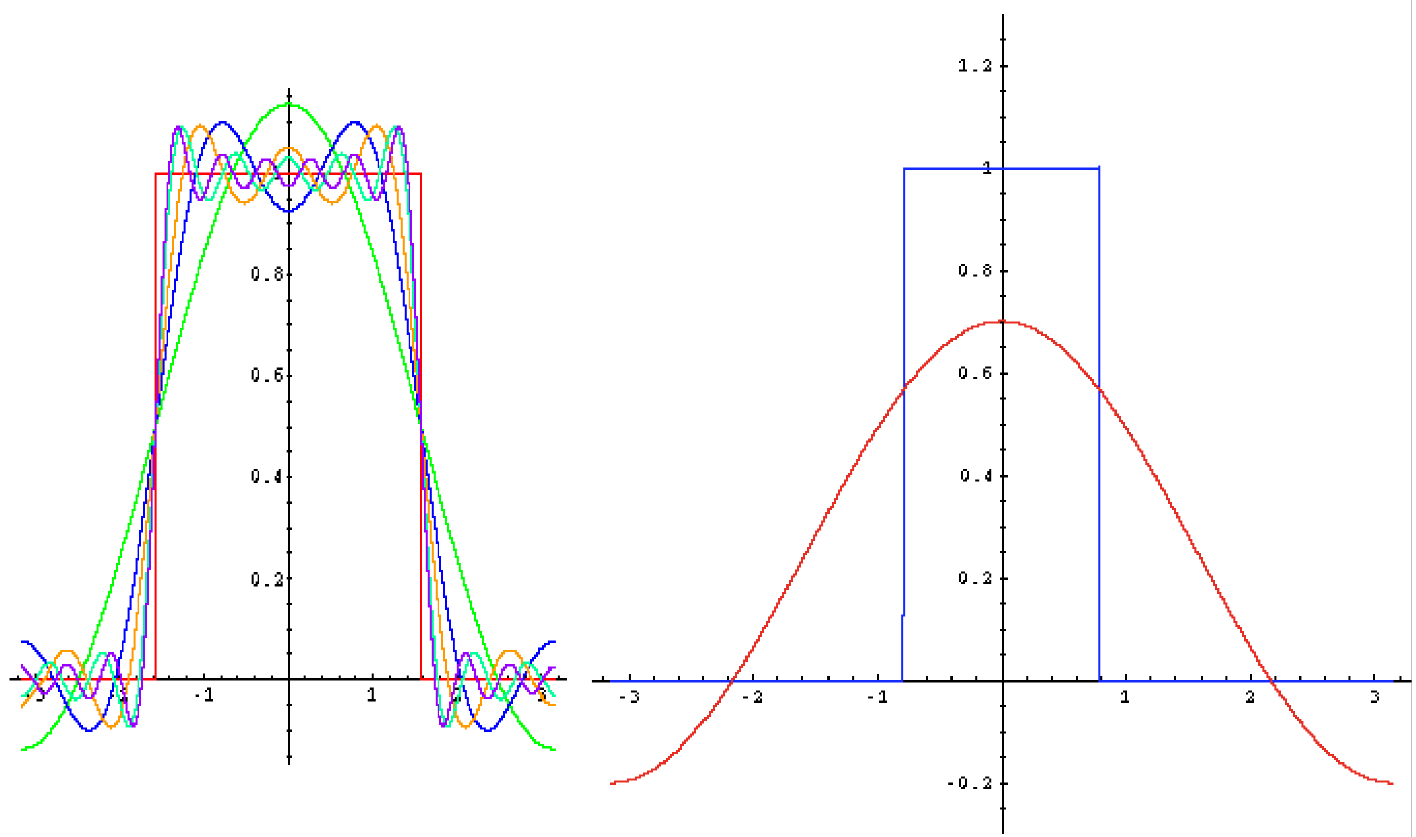
A slide from Max's talk earlier this week

Wait – what's a basis function again?



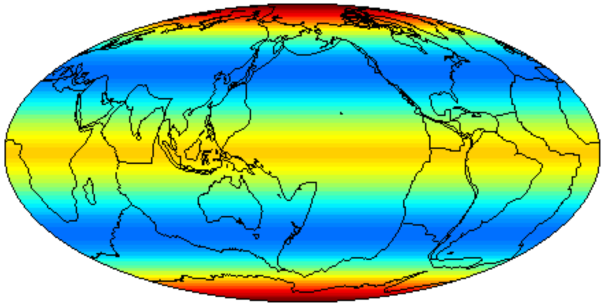
In 2D cartesian co-ordinates

Building a boxcar from cosines

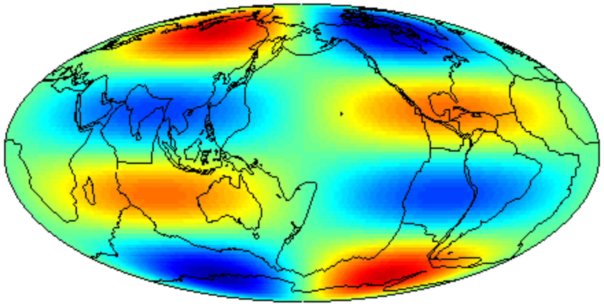


Fourier images from http://www.math.harvard.edu/archive/21b_fall_03/fourier/index.html

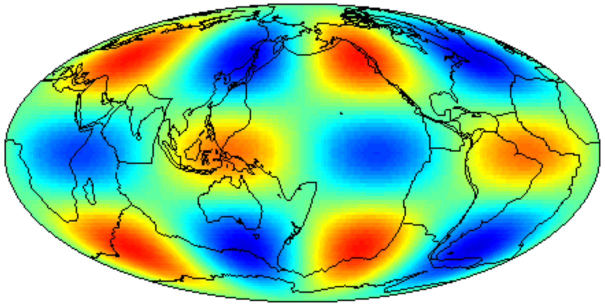
Spherical harmonics



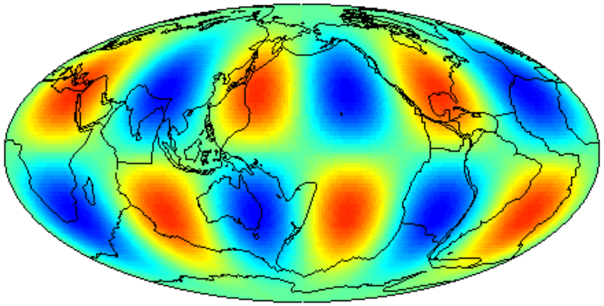
$Y_{4,0}(\theta,\phi)$



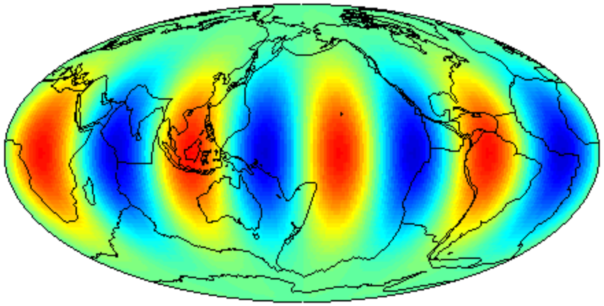
$Y_{4,1}(\theta,\phi)$



$Y_{4,2}(\theta,\phi)$



$Y_{4,3}(\theta,\phi)$



$Y_{4,4}(\theta,\phi)$

Plotted are spherical harmonics for l=4, m=0,1,2,3,4

There is lots of lovely matlab code on Frederik Simons' webpage (frederik.net) for spherical harmonics and other geophysical applications, some of which was used to make these figures

What are normal modes?

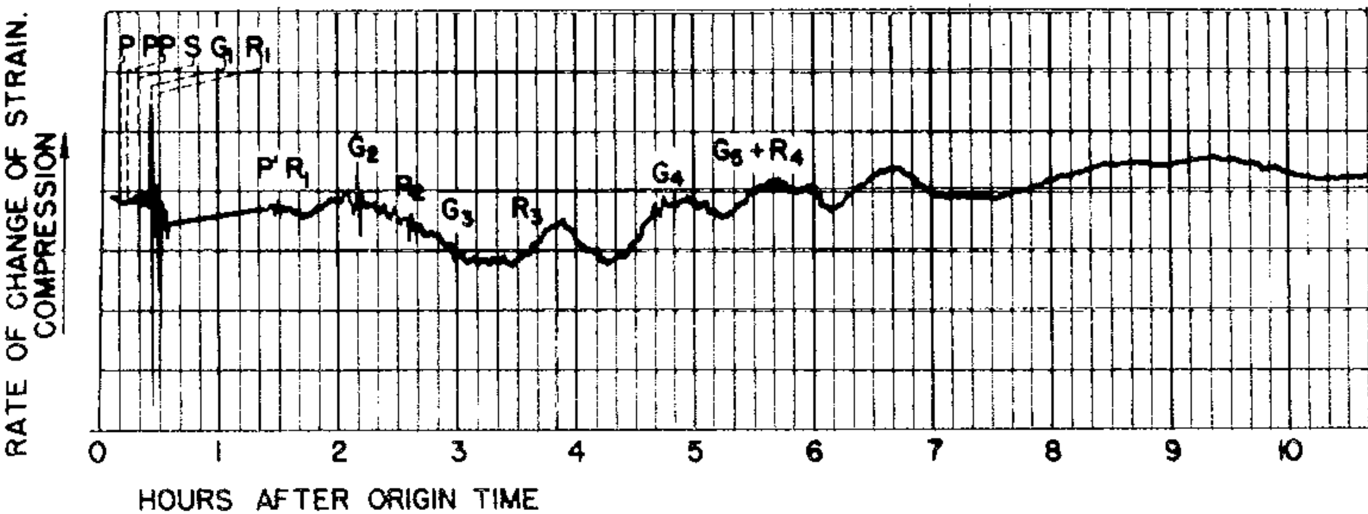
So what are Earth's normal modes?

- Whole Earth oscillations
- Other planets and moons will also undergo free oscillations and the oscillations of the sun are studied by astrophysicists: 'helioseismology'. If we have time there will be some planetary mode results shown at the end of today.
- Normally excited by an earthquake; may also be excited by atmosphere-solid Earth coupling (continuous hum).

Why should I care?

- They are used in models which underpin a lot of what we do!
- Normal modes can tell us about the gross properties of the Earth!
- They care about **density** as well as seismic velocities. (See also Guy's lecture tomorrow)

A first observation?



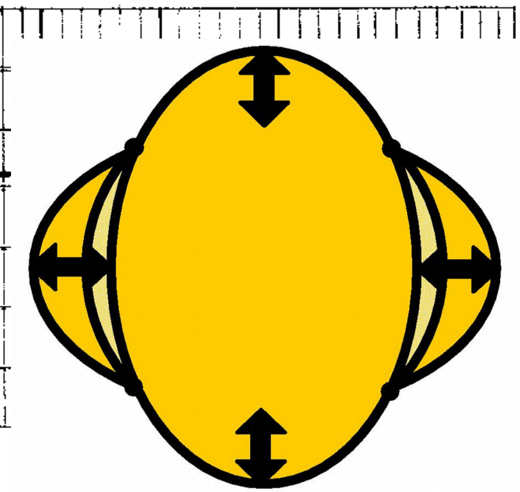


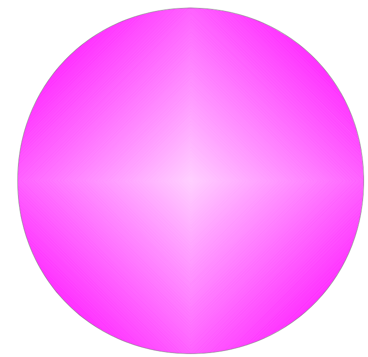
Fig. 8--Seismogram of Kamchatka earthquake, November 4, 1952, recorded by Benioff strain seismograph at Pasadena, drafted with 22 fold reduced recording rate

The origin of these ultra-long-period oscillations is not clear. The very long-period dilatation beginning at 30 min may represent a readjustment of the world crustal strain pattern in response to the release of a compressional strain on the Kamchatka fault. The 57-min and 100-min oscillations may represent free vibrations of the Earth as a whole or of the mantle as a whole. A possible mode for the 57-min vibration is one in which the Earth takes on the forms of a prolate and oblate ellipsoid alternately. In Figure 8 the straight portion of the seismogram between R_1 and $P'R_1$ represents the mean line position for a long train of Rayleigh waves on the original seismogram with periods too short (approximately 20 sec) to be resolved in the drafted copy.

Benioff, Gutenburg & Richter, *Eos T. Am. Geophys. Un., 1954*

Deriving a (simple) set of normal modes

Some mathematics:



We will consider the normal modes of a homogeneous liquid sphere. The sphere will have a constant density, ρ, a constant bulk modulus, κ, and a radius r_0.

If there are small perturbations in pressure, P which cause the excitation of normal modes, the equation of motion can be written as:

$$\rho \ddot{\boldsymbol{u}} = -\nabla P$$

And we have simplified Hooke's law to

$$P = -\kappa \nabla \cdot \boldsymbol{u}$$

Because there is only the normal stress to consider in a liquid.

We then find that:

$$c^2 \nabla^2 P = \frac{\partial^2 P}{\partial t^2} \qquad c = \sqrt{\frac{\kappa}{\rho}}$$

Here we follow Aki & Richards; I also commend to you Dahlen and Tromp

Deriving a (simple) set of normal modes

A stress free boundary condition requires that at the surface, P=0 for all times. This is clearly easiest to consider in spherical polar coordinates.

$$\nabla^2 P = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial P}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 P}{\partial \phi^2}\right)$$

We will look for a separable solution to this equation, where

$$P = P(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)e^{(i\omega t)}$$

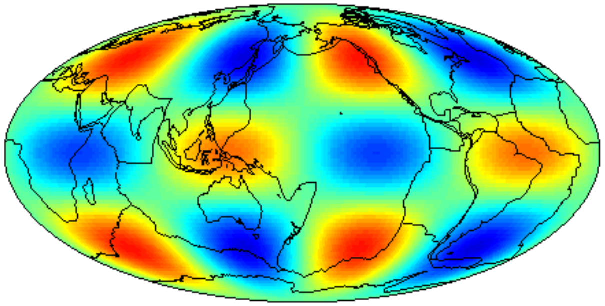
The solution to the angular terms of this equation can be written in terms of spherical harmonics (solution not derived here):

$$\Theta(\theta)\Phi(\phi) = Y_l^m(\theta,\phi) = (-1)^m \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

Where we have written the spherical harmonics in terms of the Legendre polynomials:

Deriving a (simple) set of normal modes

The spherical harmonics are the images on screen a few slides back:



We have found the angular dependence of our solutions: $P = R(r)\Theta(\theta)\Phi(\phi)e^{(i\omega t)}$
So the remaining term to solve for is the radial term:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\left[\frac{\omega^2}{c^2}-\frac{l(l+1)}{r^2}\right]R=0$$

Which contains a dependence on l, but not on m.

As we also want our solutions to have no singularities, it turns out that the radial dependence of P is given by

$$R \propto j_l\left(\frac{\omega r}{c}\right)$$

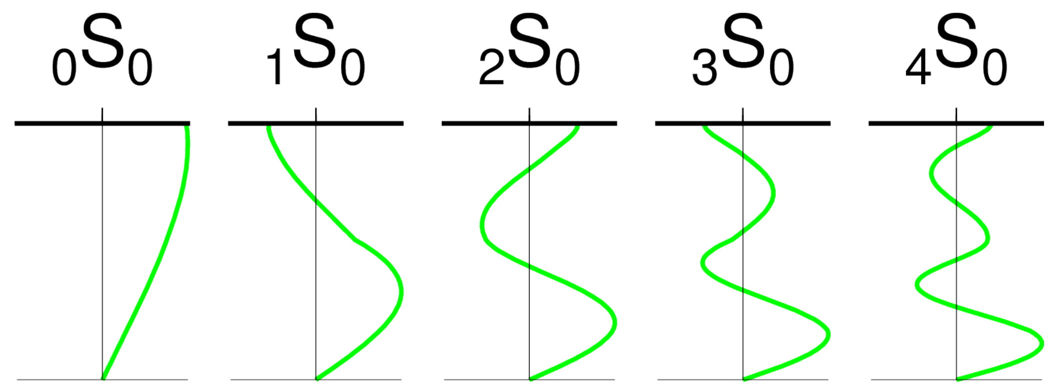
where

$$j_l(x) = x^l\left(\frac{-1}{x}\frac{d}{dx}\right)^l\left(\frac{(\sin x)}{x}\right)$$

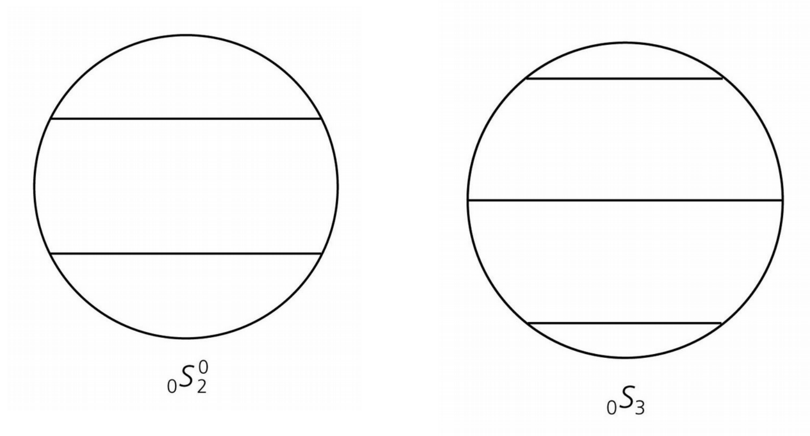
are Bessel functions.

Deriving a (simple) set of normal modes

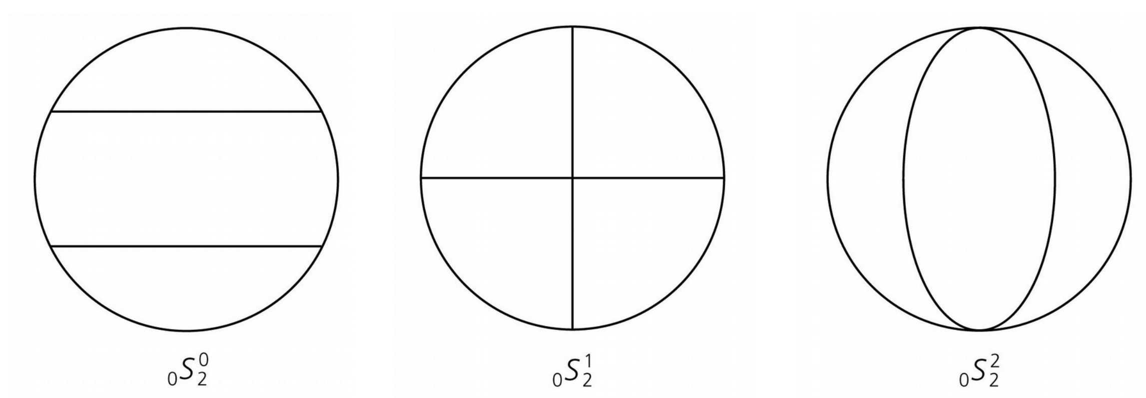
All of these letters! n, l and m are:



n – overtone number or radial order – think of this as the number of zeros along the radius of the earth for a mode with l=0



l – angular order – think of this as the number of zeros across the surface of the sphere the radius of the earth for a mode



m – azimuthal order – think of this as telling us about how those lines zeros are ordered on the surface of the sphere

lower 5 figures from Stein & Wysession, p 106

Deriving a (simple) set of normal modes

Then for l=0

$$R(r) \propto \frac{c}{r\omega} \sin\left(\frac{r\omega}{c}\right)$$

For l =1

$$R(r) \propto \frac{c^2}{r^2\omega^2} \sin\left(\frac{r\omega}{c}\right) - \frac{c}{r\omega} \cos\left(\frac{r\omega}{c}\right)$$

And so on.

We still need the stress free boundary condition to be satisfied: $P(r_0)=0$

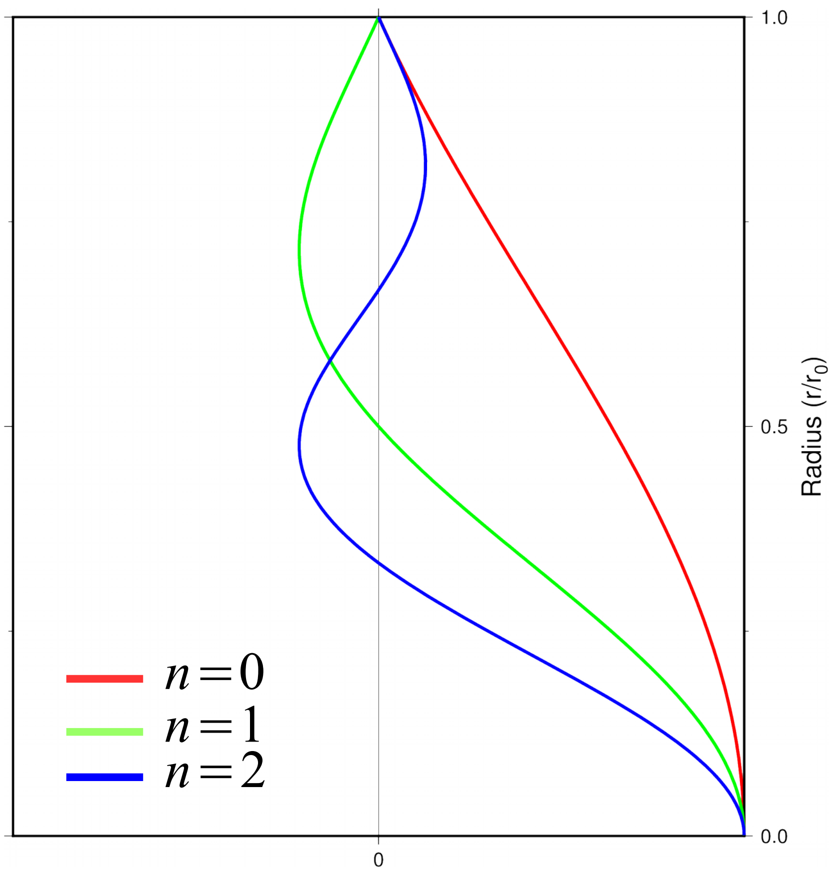
This places restrictions on the allowable values of ω:

$${}_n\omega_0 = \frac{(n+1)\pi c}{r_0}$$

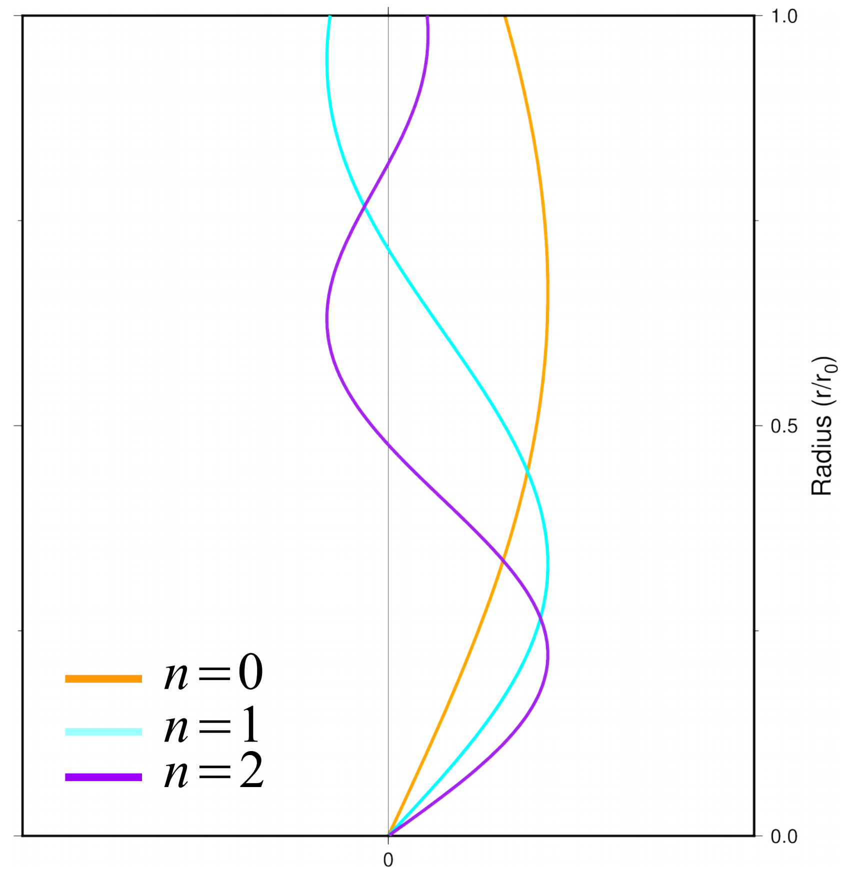
Where we have now found the frequencies of the radial modes for the homogeneous, liquid sphere.

Radial variations

The radial functions of the pressure term can be plotted:



$_nR_0(r)$



$_nR_1(r)$

$$P = P(r, \theta, \phi, t) = R(r)\Theta(\theta)\Phi(\phi)e^{(i\omega t)}$$

Normal modes!

These normal modes are clearly much less complex than those of the Earth, but already have the properties of Earth's normal modes.

Notice as well that they are degenerate – the frequencies of oscillation are not dependent on m, but only on n and l.

The modes form a complete, orthogonal, basis set. We have considered a fluid, homogeneous sphere, without taking into account self-gravitation. We have also not considered attenuation in the formalization of this problem – something which will clearly affect the Earth's normal modes. Finally we have not talked about how the initial pressure perturbations are generated.

Now, we can write any change in the pressure in the material in sphere in terms of a sum of our normal modes:

$$P = \sum_{n,l,m} {}_n A_l^m R(r) Y_l^m(\theta, \phi) \exp(i \, {}_n\omega_l^m t)$$

And the corresponding displacements could also be obtained.

More realistic normal modes

The same type of modeling of normal modes can be applied to the more complex structure we know to exist on Earth. Just as we did with body waves, and surface waves, we can break the motions of the earth into two types – toroidal and spheroidal. The P-SV motions are the counterpart of the spheroidal normal modes, and the SH motions are the toroidal modes (which were forbidden in the liquid planet we just considered).

We will use three different vector spherical harmonics:

$$\boldsymbol{R}_l^m(\theta,\phi)=Y_l^m\hat{\boldsymbol{r}} \qquad \boldsymbol{S}_l^m(\theta,\phi)=\frac{1}{\sqrt{l(l+1)}}\left(\frac{\partial Y_l^m}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\boldsymbol{\phi}}\right)$$

$$\boldsymbol{T}_l^m(\theta,\phi)=\frac{1}{\sqrt{l(l+1)}}\left(\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\hat{\boldsymbol{\theta}}-\frac{\partial Y_l^m}{\partial\theta}\hat{\boldsymbol{\phi}}\right)$$

Let us write the displacement of a normal mode with a particular n, l and m as:

$$\left[{}_nU_l(r)\boldsymbol{R}_l^m(\theta,\phi)+{}_nV_l(r)\boldsymbol{S}_l^m(\theta,\phi)+{}_nW_l(r)\boldsymbol{T}_l^m(\theta,\phi)\right]\exp(-i\,{}_n\omega_l t)$$

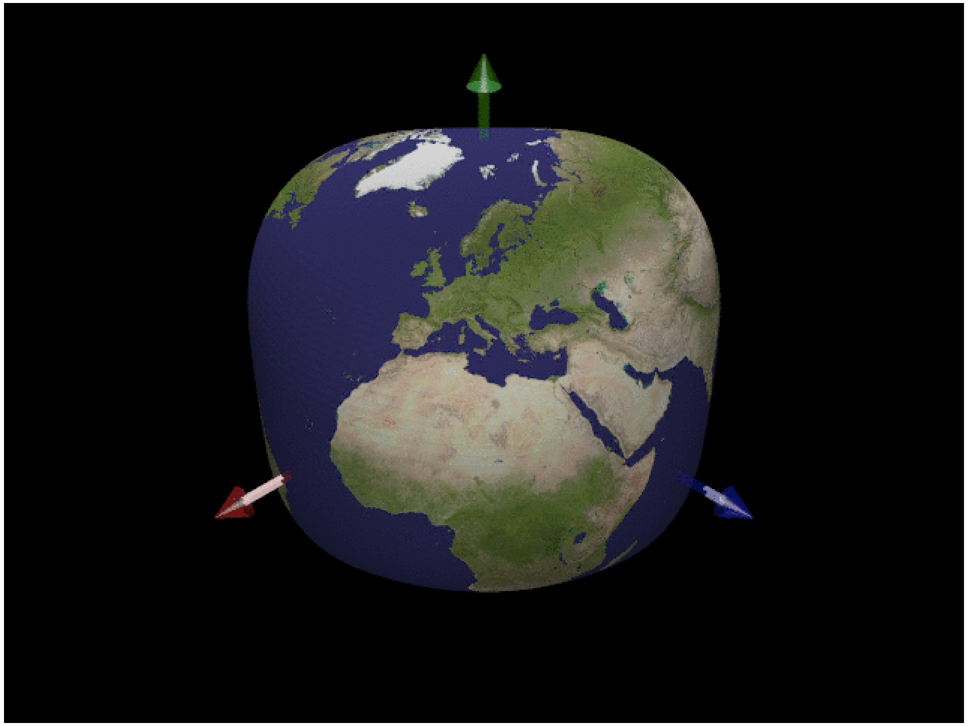
we could do the same for the traction, and then use the elastic tensor to relate the two.

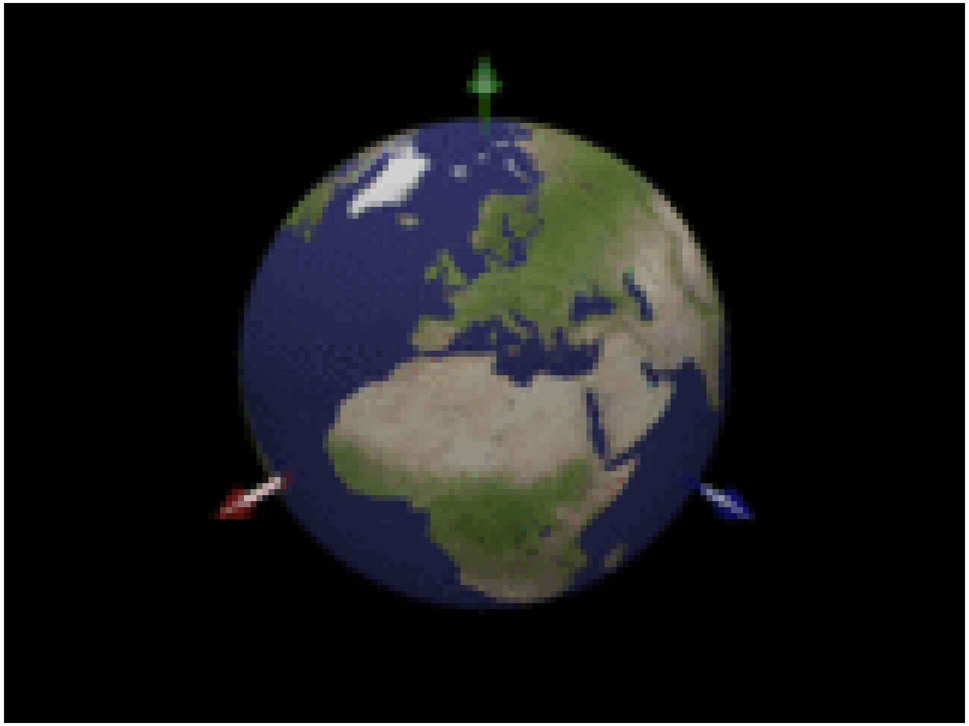
Two types of normal modes

Doing some rearranging, we would find we have split the modes into two parts – the one which contains motion described by the $\boldsymbol{R}_l^m$ and $\boldsymbol{S}_l^m$ vector fields, and one whose motion is described by the $\boldsymbol{T}_l^m$ vector field.

The first of these, the spheroidal modes, have the radial component of $\nabla \times \boldsymbol{u}$ equal to zero.

The second of these, the toroidal modes have both $\nabla \cdot \boldsymbol{u} = 0$ and $u_r = 0$





Normal modes!

A quick reminder of the terminology we've got so far:

n = radial order

l = angular order

m = azimuthal order

Labeling modes:

$_{n}S_{l}$

Spheroidal mode

$_{n}T_{l}$

Toroidal mode

More realistic normal modes

We can now consider the more complex problem of the self-gravitating Earth.

Gravity is the reason that planetary bodies are (roughly) spherical.

We would start with Poisson's equation:

$$\nabla^2 V_0 = -4\pi G \rho_0$$

Where V_0 is the gravitational potential and is a function of r, G is the gravitational constant and ρ_0 is the density and also a function of r.

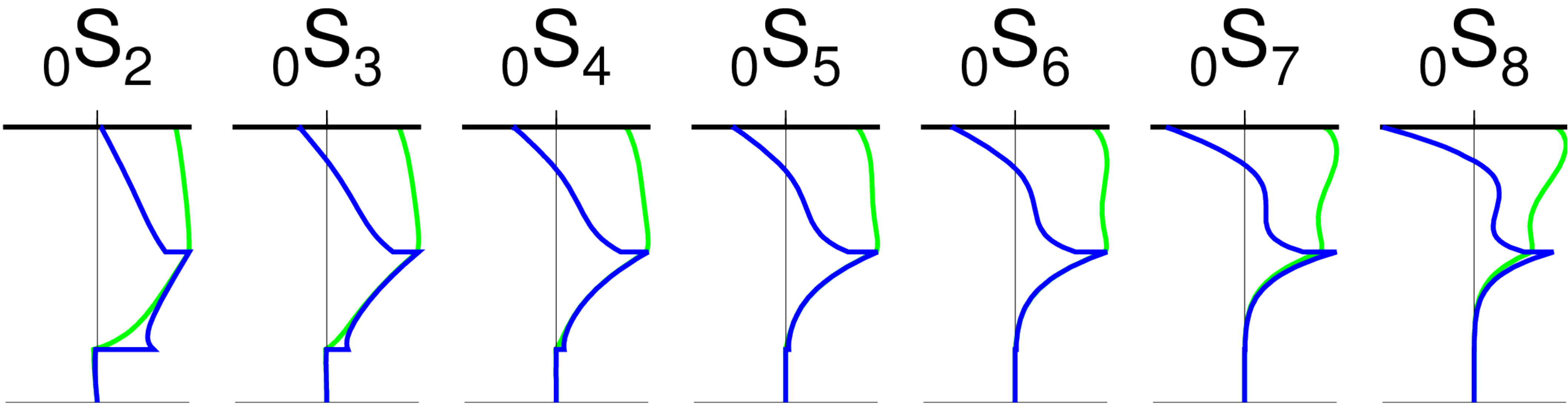
Where would this lead us?

To a more (but not perfectly) realistic set of normal modes!

Displacements of normal modes

These equations can also be written in matrix form and solved for the ${}_n\omega_l$ and the displacements which are given by U(r) and V(r).

Gravity does not affect the toroidal modes, so the matrix equation for W(r) and T(r) is simpler.



{ remember displacement looks like $\left[{}_nU_l(r)\boldsymbol{R}_l^m(\theta,\phi)+{}_nV_l(r)\boldsymbol{S}_l^m(\theta,\phi)+{}_nW_l(r)\boldsymbol{T}_l^m(\theta,\phi)\right]\exp(-i\,{}_n\omega_l t)$ }

Displacements of normal modes

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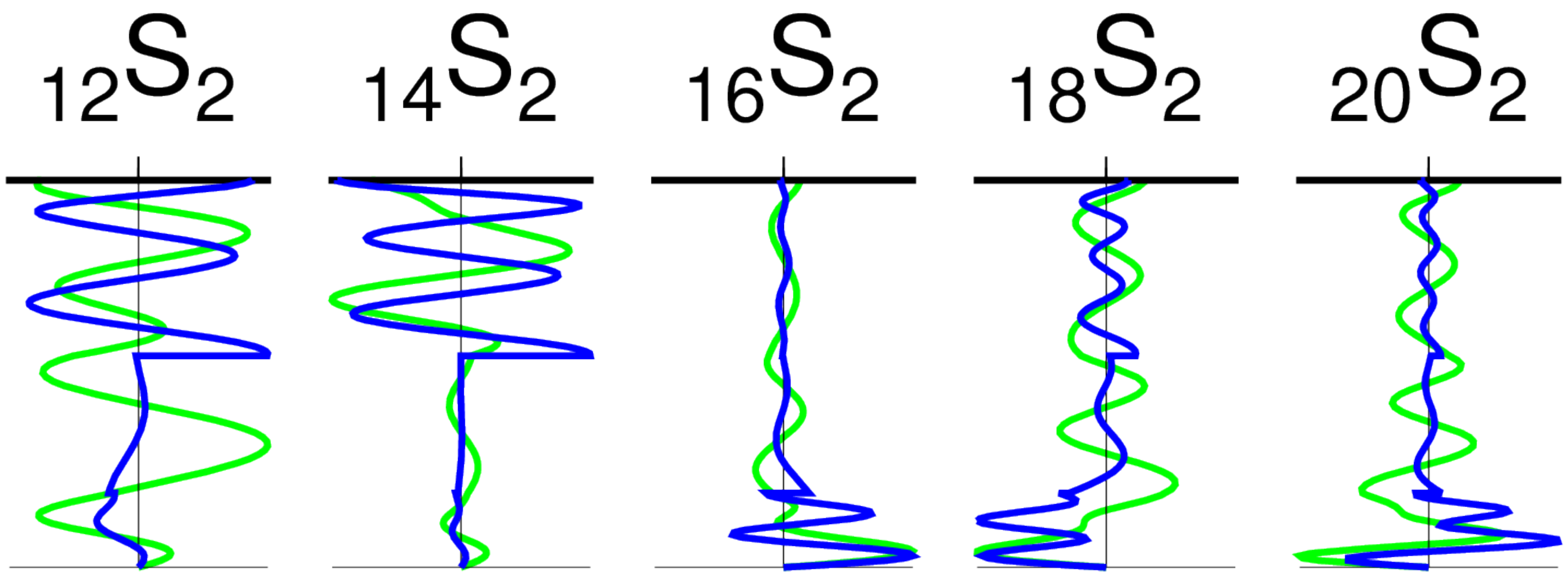
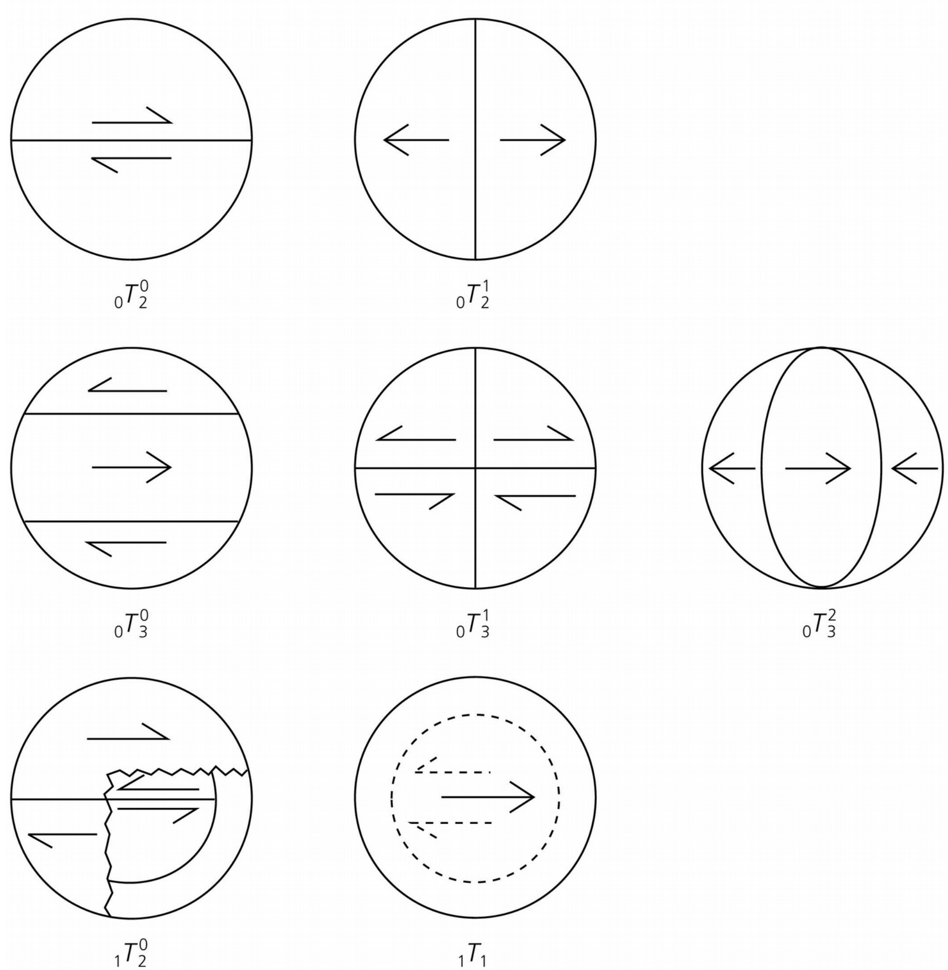


Lateral variations of the modes

The equations for the normal modes have 2l+1 degeneracy – there are 2l+1 singlets which are excited as part of every mode, corresponding to different values of m.

For a toroidal mode, the different singlets have motions with l-1 nodal planes on the surface:

Figure 2.9-6: Examples of the displacements for several torsional modes.



From Stein & Wysession, p 105

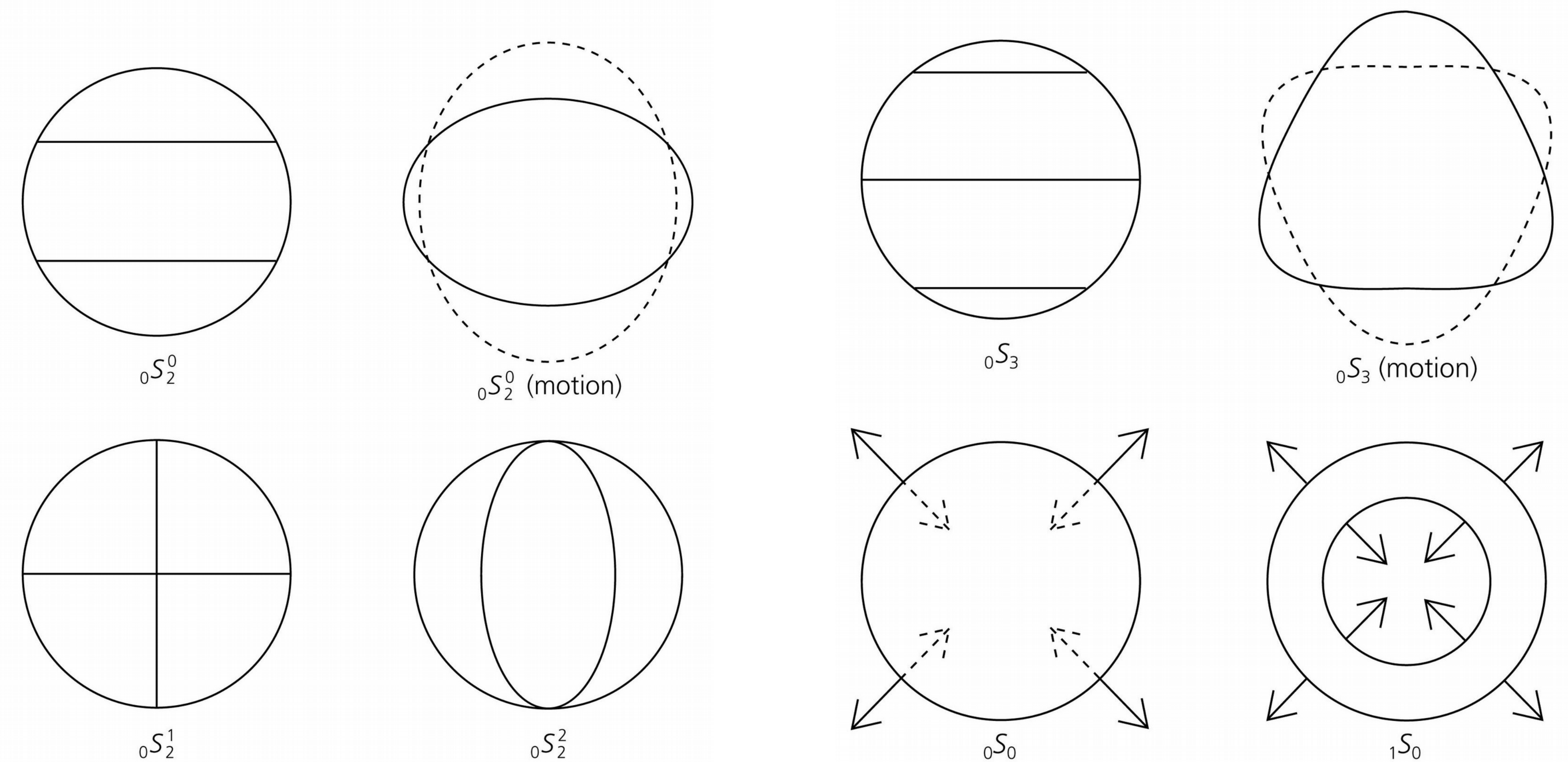
Lateral variations of the modes

The equations for the normal modes have 2l+1 degeneracy – there are 2l+1 singlets which are excited as part of every mode, corresponding to different values of m.

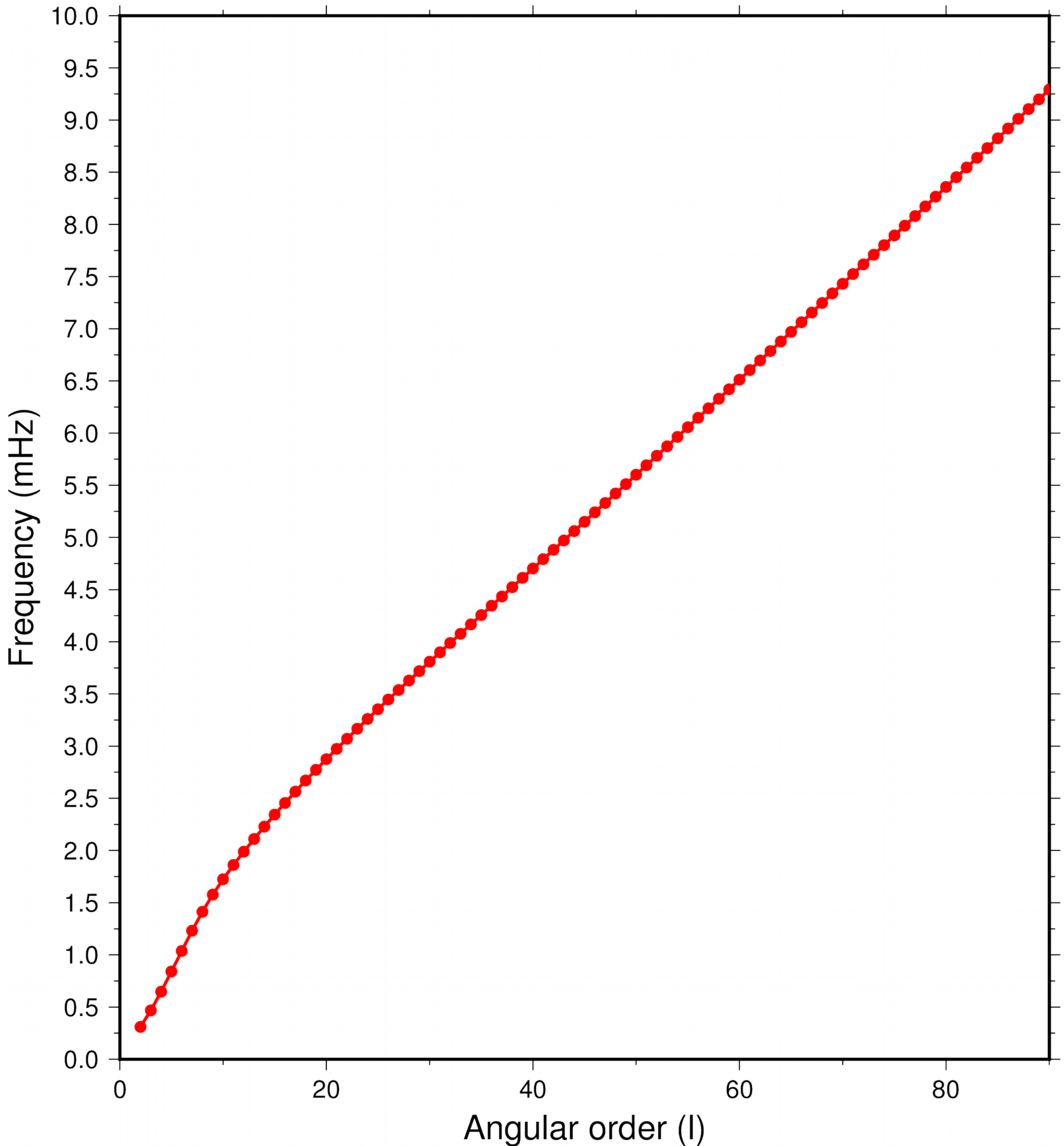
For a spheroidal mode mode, the different singlets have motions with l-1 nodal planes on the surface:

Figure 2.9-7: Examples of the displacements for several spheroidal modes.

From Stein & Wysession, p 106

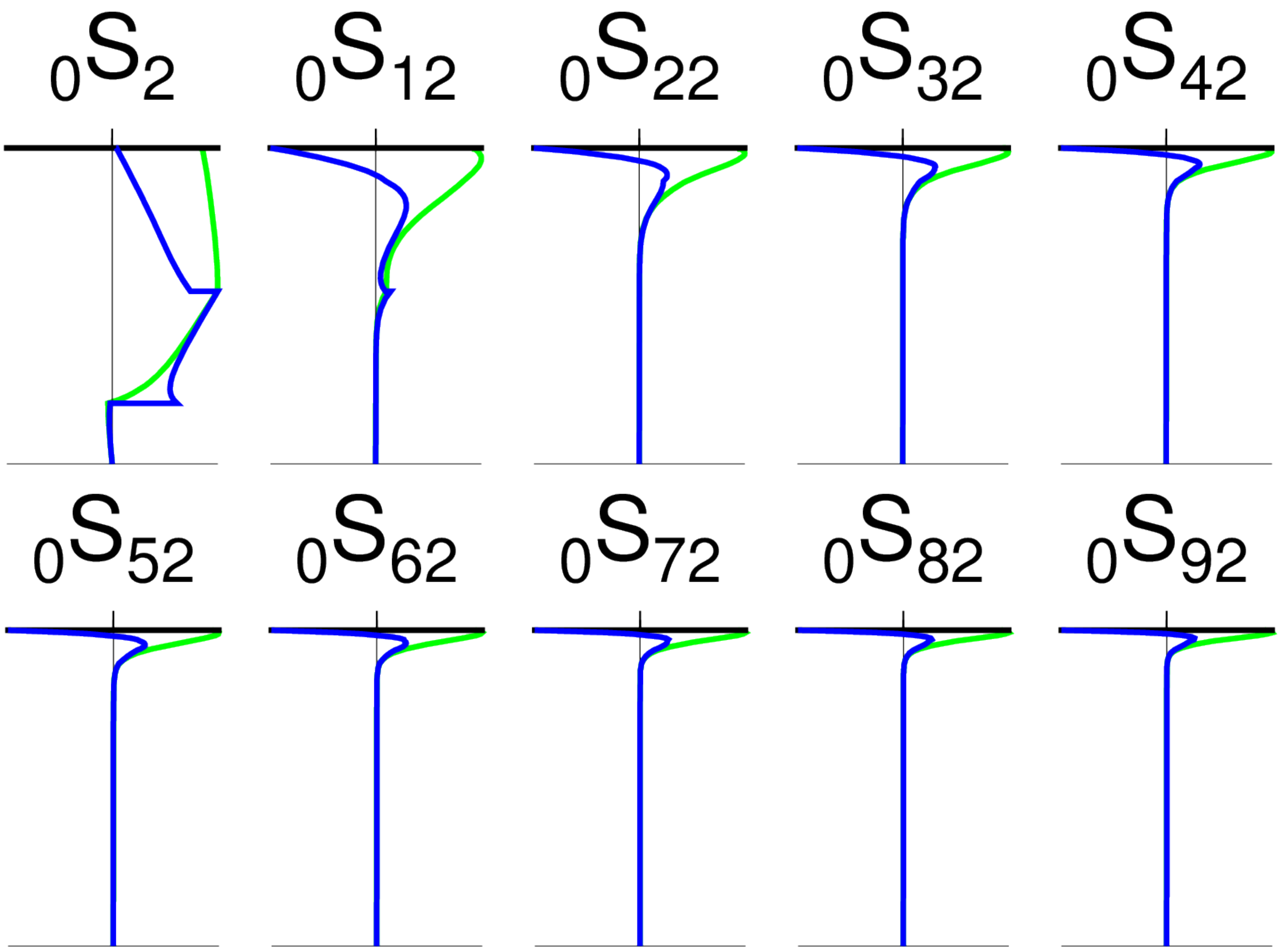


Normal mode branches



n=0;
l increasing

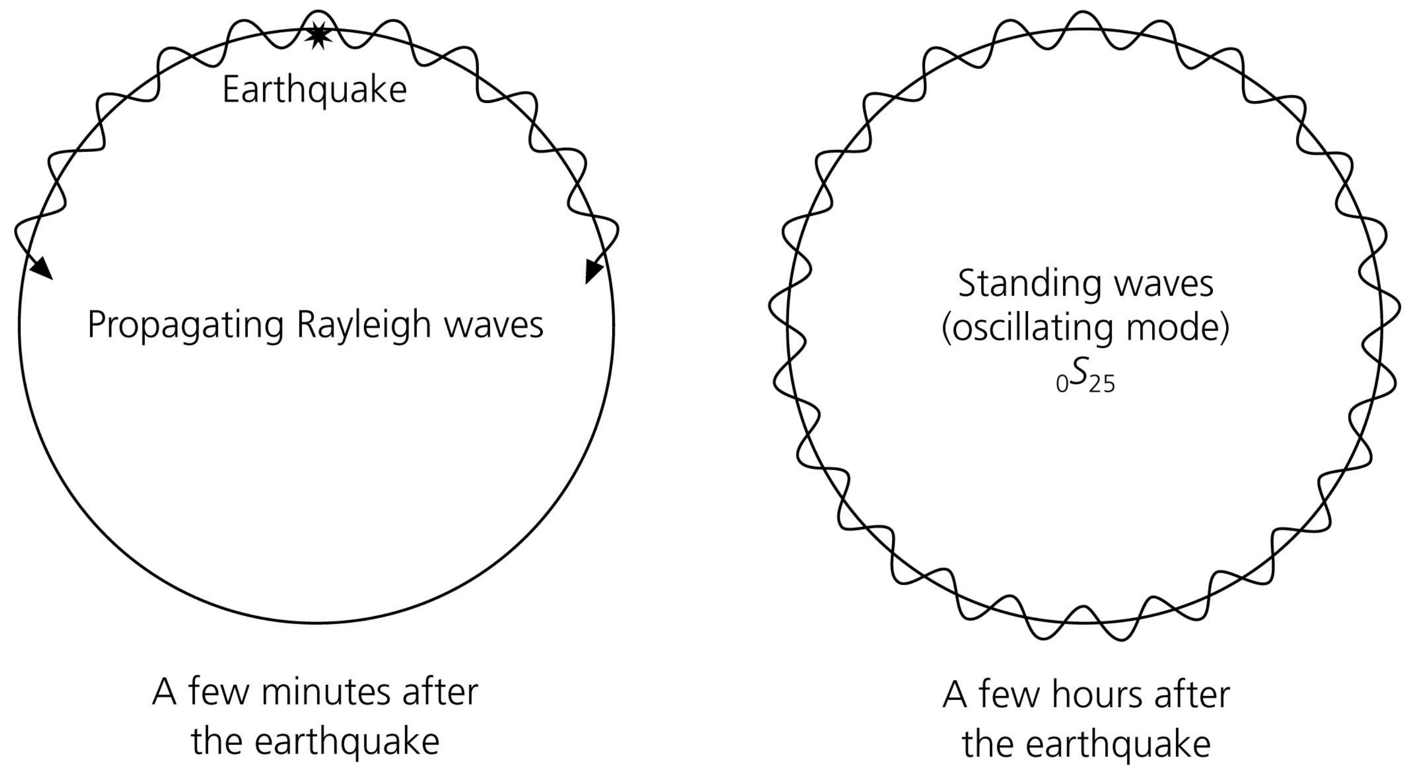
Normal mode branches



Normal mode : surface wave equivalence

There is an equivalence between surface waves and normal modes:

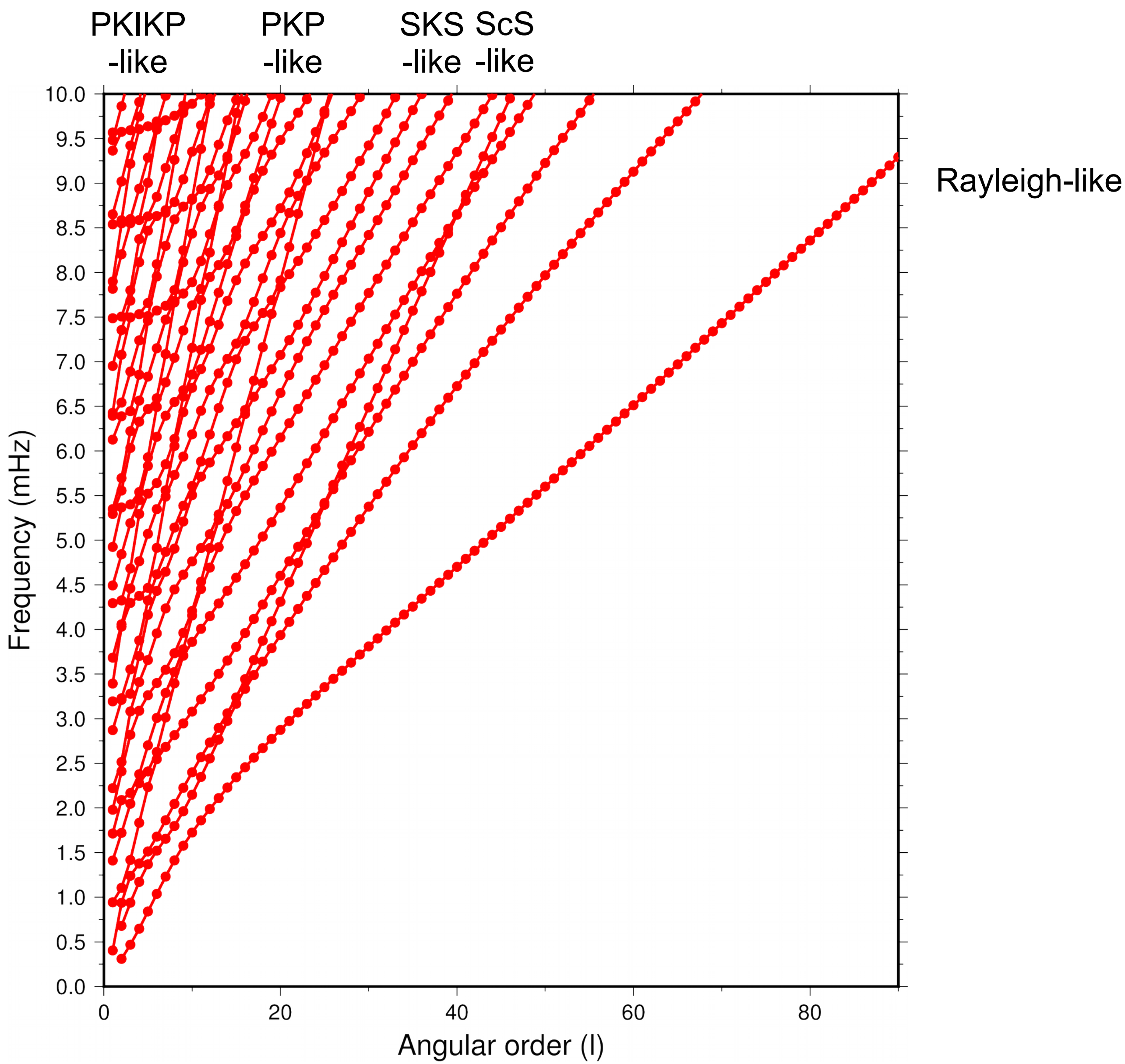
Figure 2.9-8: Cartoon of the equivalence of surface waves and normal modes.



From Stein & Wysession, p 107

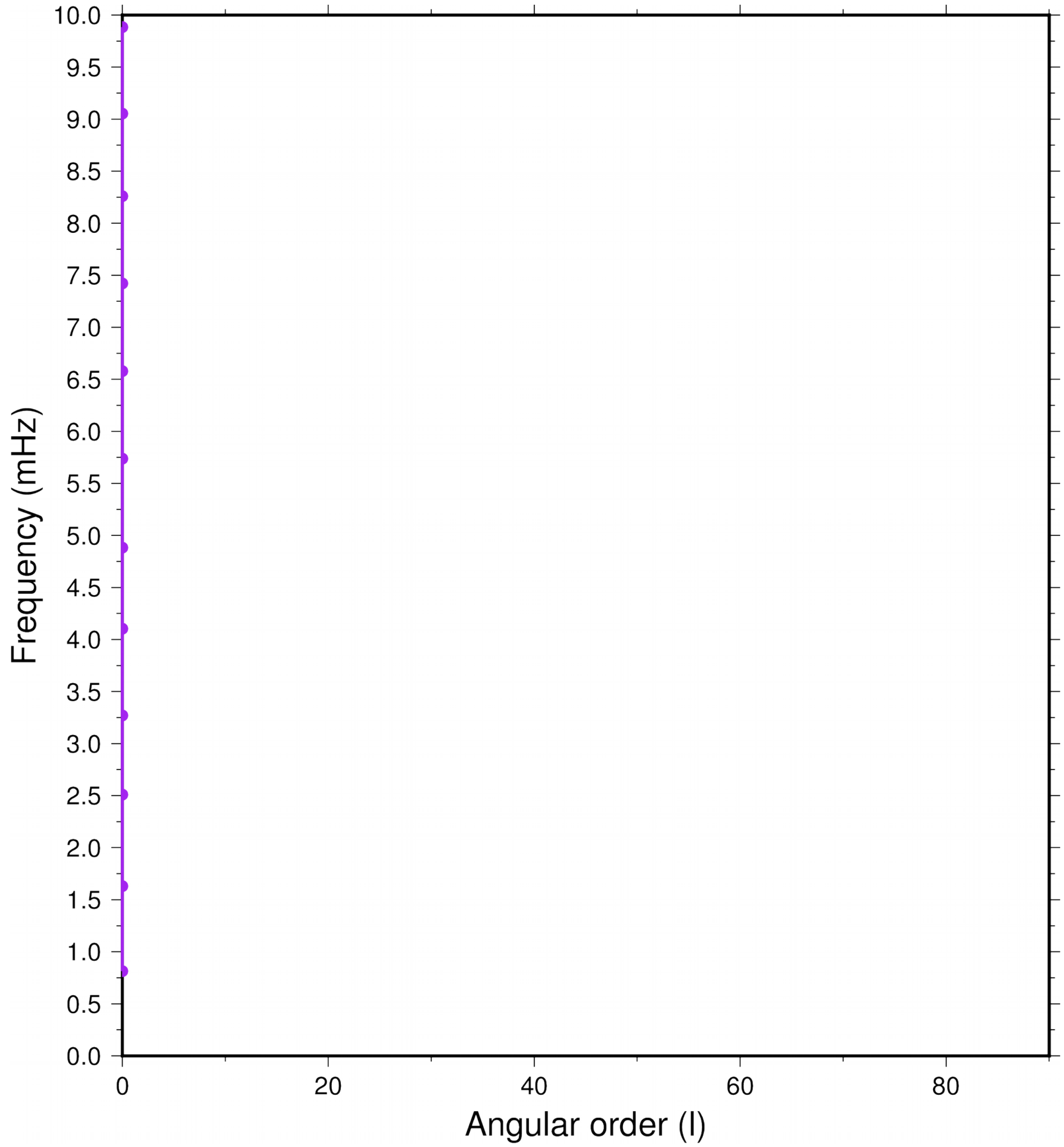
A surface wave which has $l+\frac{1}{2}$ wavelengths equivalent to the circumference of the Earth can be compared to the mode with angular order l. The surface wave will then move with a horizontal phase velocity of $c_x = \frac{{}_n\omega_l}{|\boldsymbol{k}_x|} = \frac{{}_n\omega_l a}{l+1/2}$

Normal mode branches



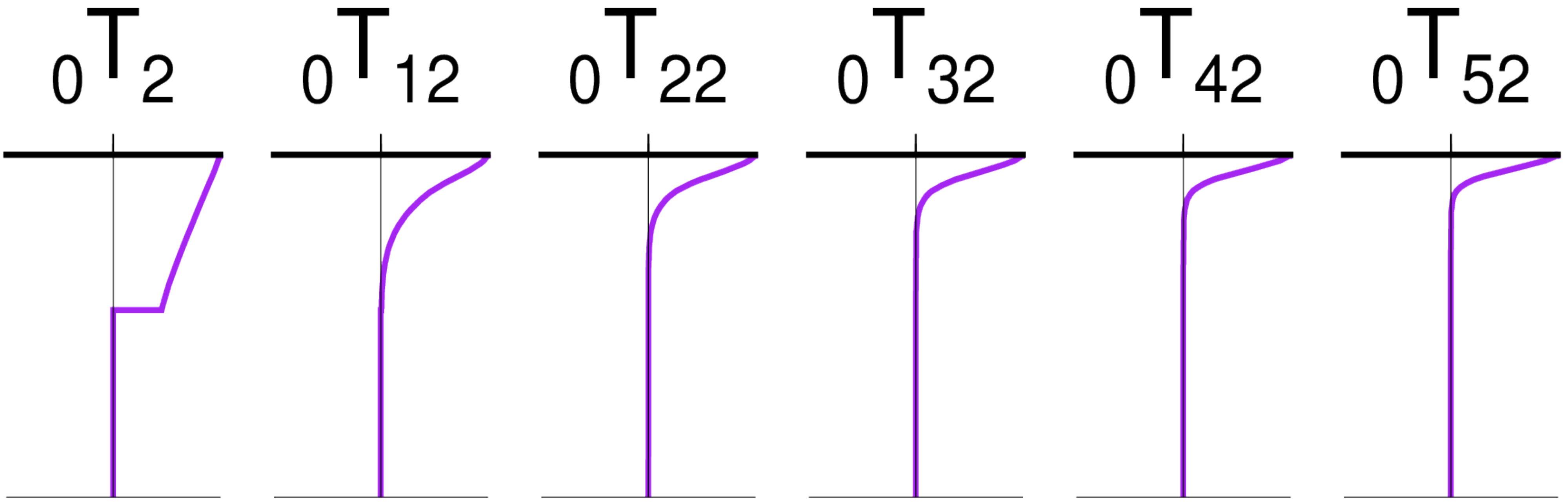
Labels after Lay & Wallace (1995)

Radial normal modes



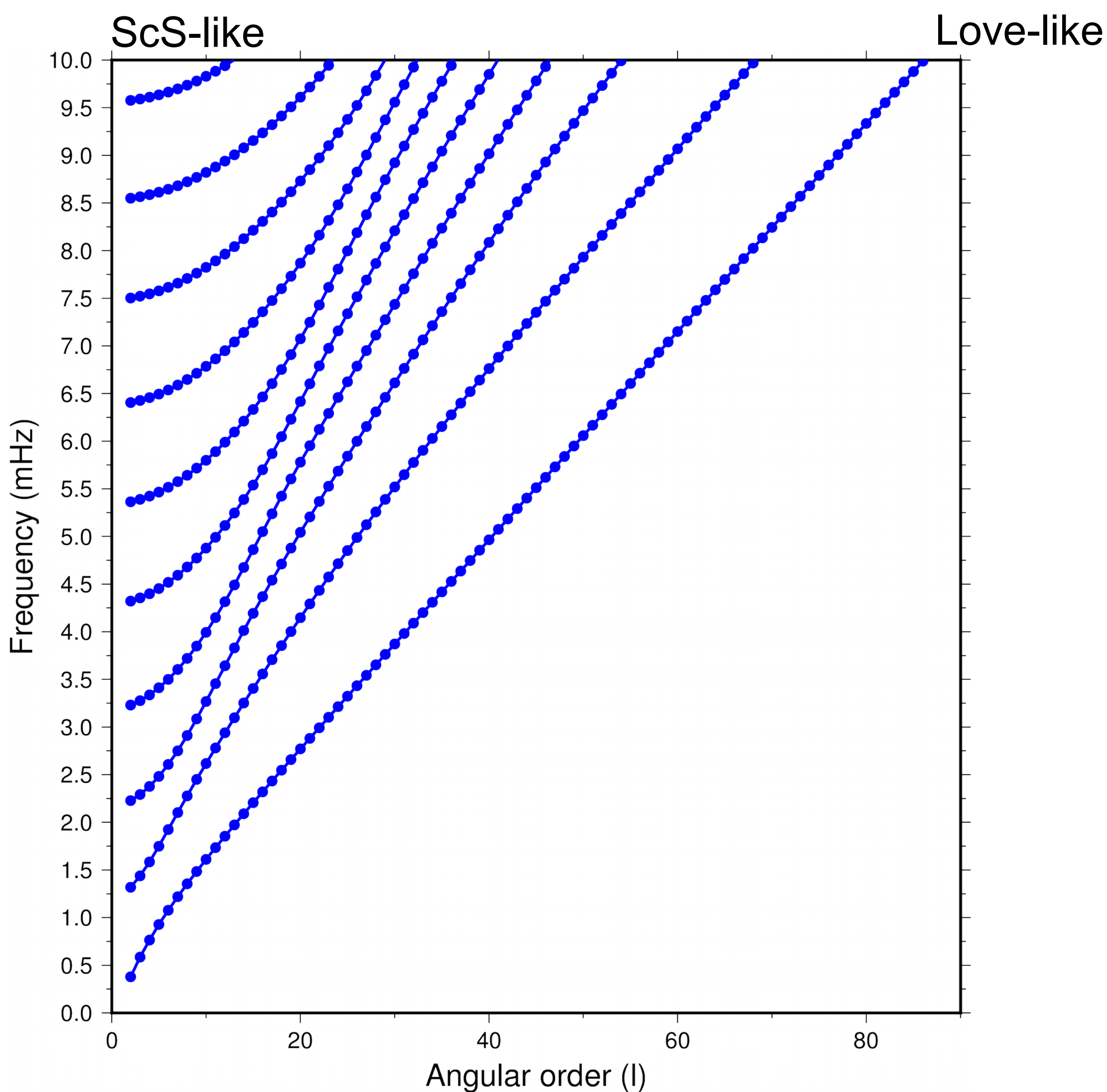
l=0;
n increasing

Normal mode branches

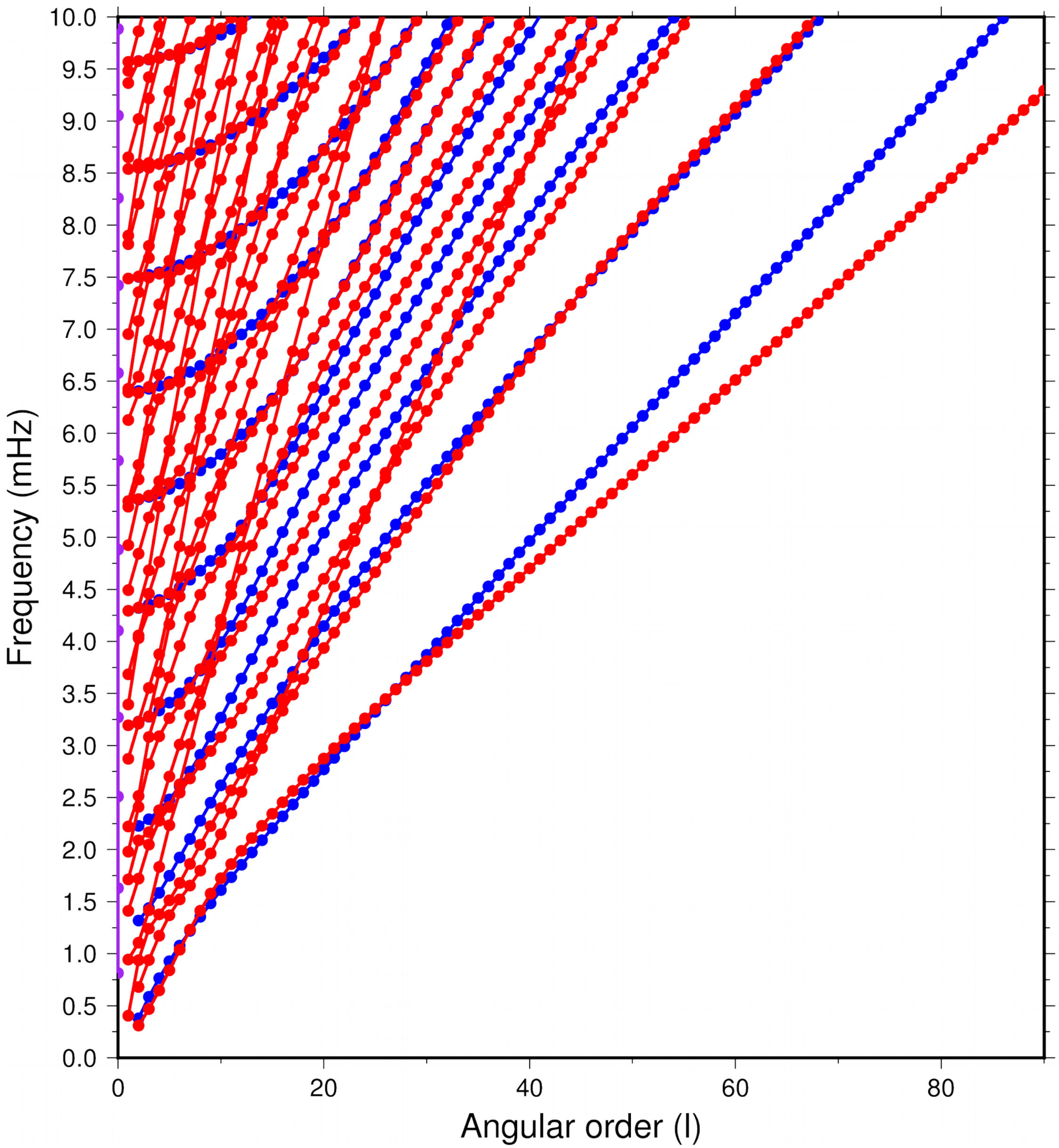


Normal mode branches

For toroidal modes:



Normal mode branches



Normal mode decay

With no attenuation in the Earth, what would happen to normal modes generated by an earthquake?

Attenuation, or anelasticity in the earth causes a mode to decay over time. It is *one* of the reasons a mode will appear as a broadened peak in a normal mode spectrum

$$e^{i\,{}_n\omega_l^m t}\, e^{\frac{-\,{}_n\omega_l^m t}{2\,{}_nQ_l^m}}$$

A first observation

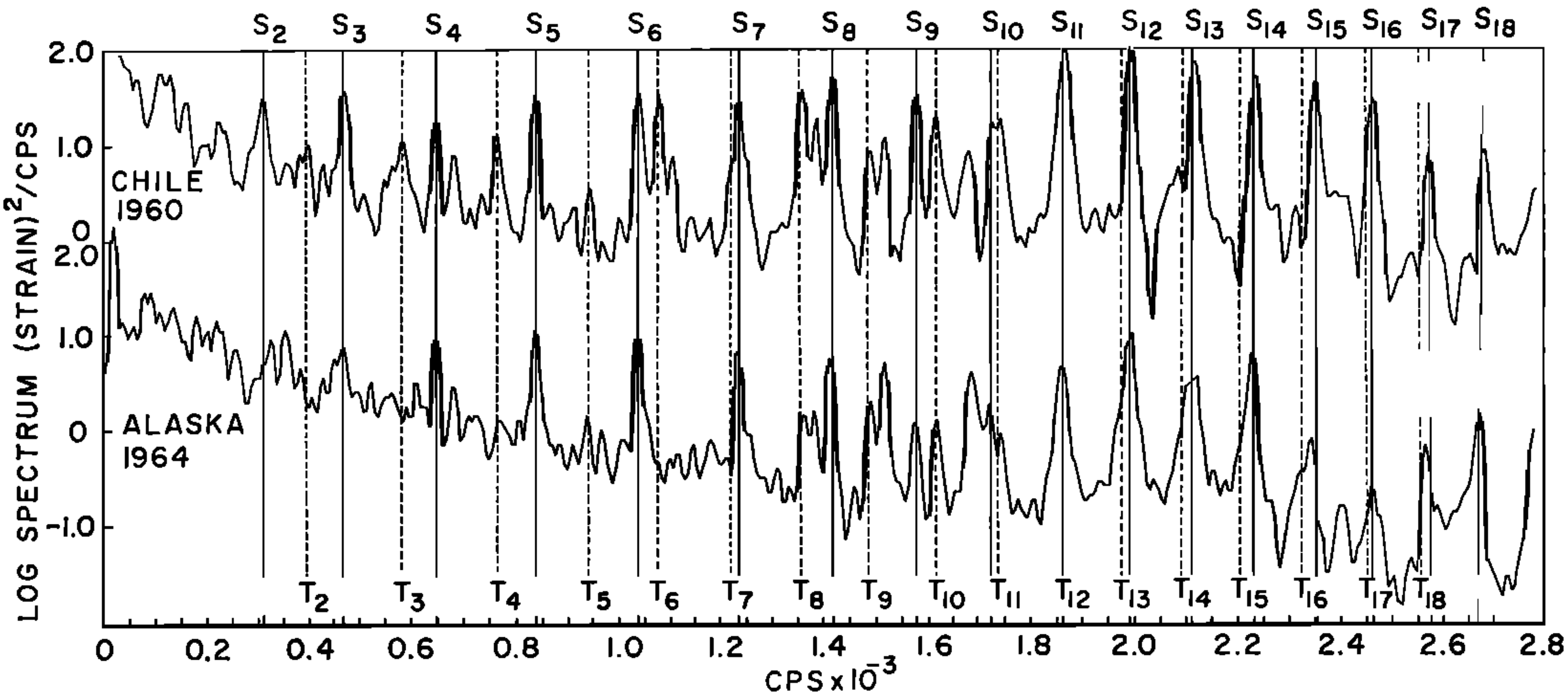


Fig. 4. Comparison of Chilean and Alaskan earthquake. Record length, 7854 min beginning 285 min after origin time for both events; sample interval, 3 min; bandwidth, $180{,}000^{-1}$ cps.

(7854 min = 130.9 hours = 5.45 days)

Smith, *J. Geophys. Res., 1966*

Normal mode spectrum

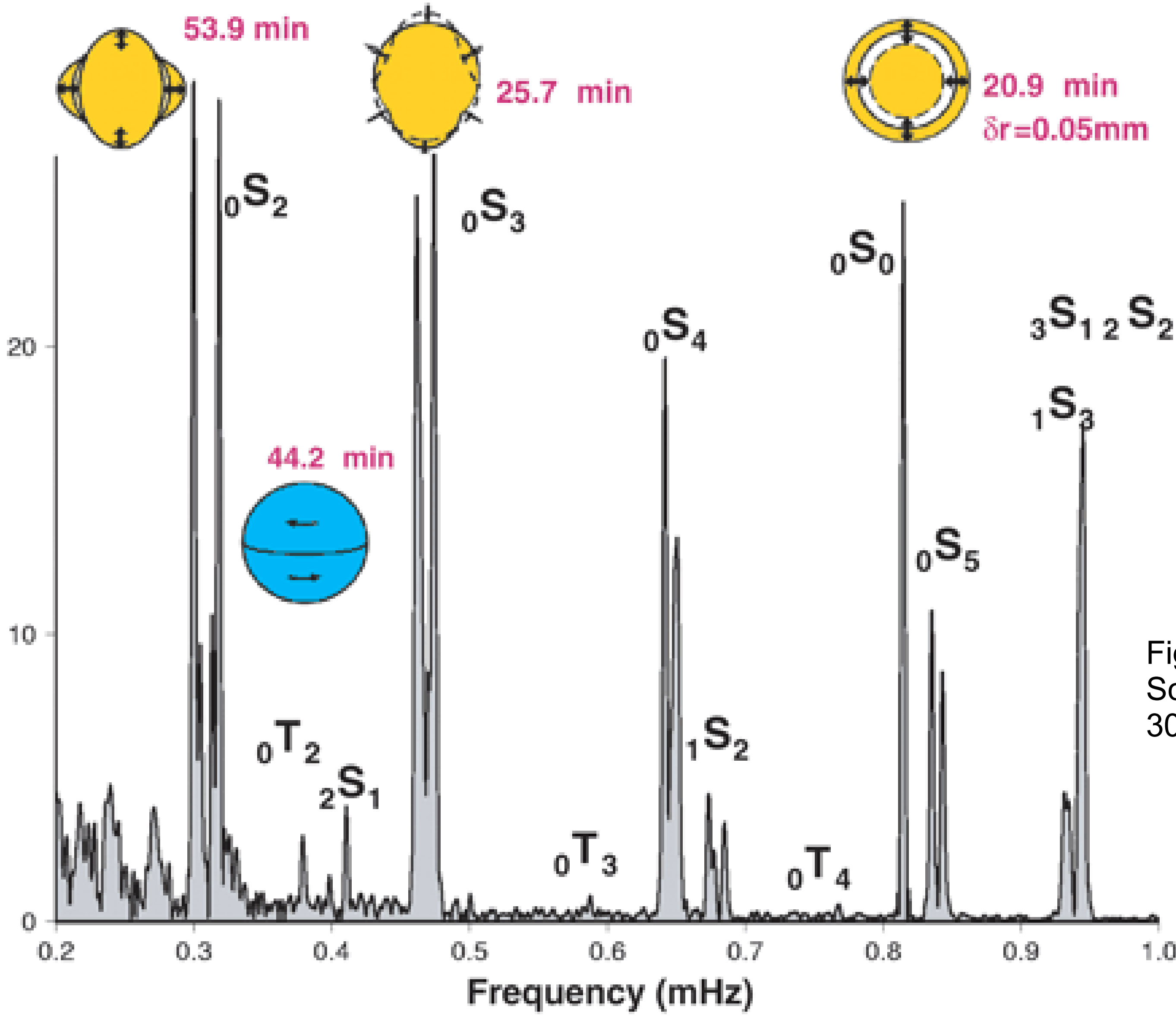


Figure from Park et al, Science 20 May 2005: 308 (5725), 1139-1144

Normal mode strengths

The frequencies of normal modes can be a powerful tool in probing the Deep Earth:

Solidity of the Inner Core of the Earth inferred from Normal Mode Observations

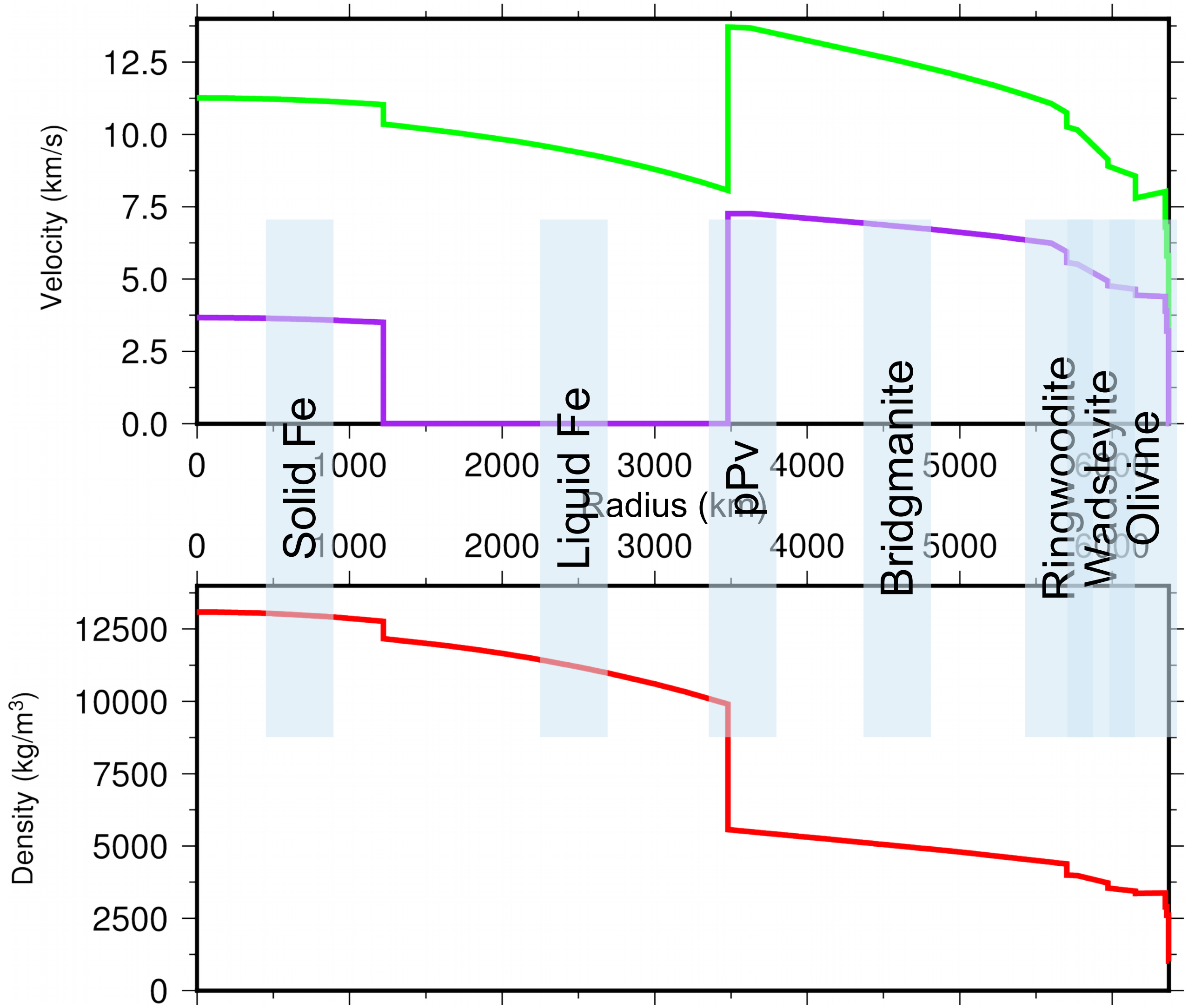
Table 1 Observed Normal Modes of the Earth sensitive to the Structure of the Inner Core

				UTD124B′—Solid inner core				UTD124B′—Liquid inner core		5.08M		HB_1	
Mode	Mean period (s)	No. of obser-vations	s.e.m. (s)	Comp. period	Rel. error (%)	Inner core energy Compr.	Inner core energy Shear	Comp. period	Rel. error (%)	Comp. period	Rel. error (%)	Comp. period	Rel. error (%)
${}_1S_0$	613.57	11	0.236	614.59	0.17	0.181	0.000	607.39	−1.02	610.06	−0.57	607.4	−1.01
${}_2S_0$	398.54	40	0.084	397.59	−0.24	0.206	0.001	392.31	−1.59	391.42	−1.81	394.0	−1.14
${}_3S_0$	305.84	7	0.129	306.00	0.05	0.233	0.003	301.36	−1.48	301.84	−1.31	300.9	−1.62
${}_4S_0$	243.59	12	0.067	243.80	0.09	0.192	0.007	241.11	−1.03	241.55	−0.84	239.9	−1.51
${}_2S_2$	904.23	21	0.487	904.43	0.02	0.001	0.080	914.94	1.17	917.80	1.50	915.1	1.20
${}_5S_2$	397.36	11	0.157	397.03	−0.09	0.015	0.102	399.93	0.67	398.20	0.21	399.1	0.44
${}_6S_1$	348.41	21	0.046	348.23	−0.05	0.068	0.011	347.10	−0.38	347.38	−0.30	346.6	−0.52
${}_7S_3$	281.37	11	0.113	281.59	0.08	0.004	0.022	282.77	0.50	283.34	0.70	282.1	0.22
${}_8S_1$	272.10	11	0.144	271.79	−0.11	0.115	0.052	271.00	−0.40	270.92	−0.43	270.5	−0.59
	Nine modes—r.m.s.				0.12				1.01		1.00		1.02

Table from Dziewonski & Gilbert, *Nature*, 1971

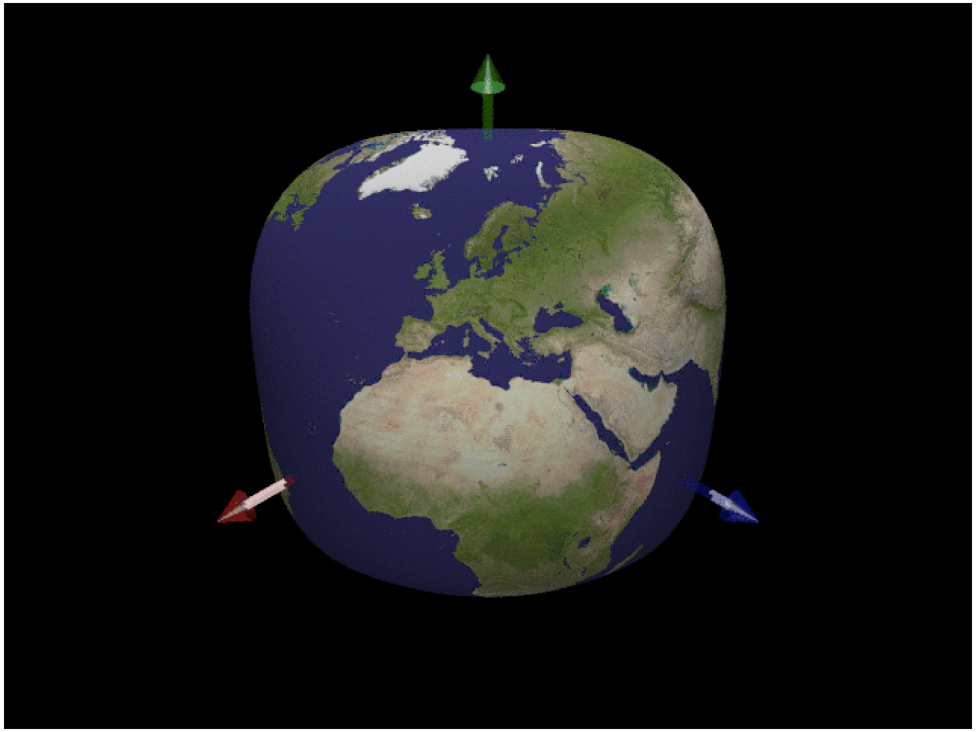
Normal mode strengths

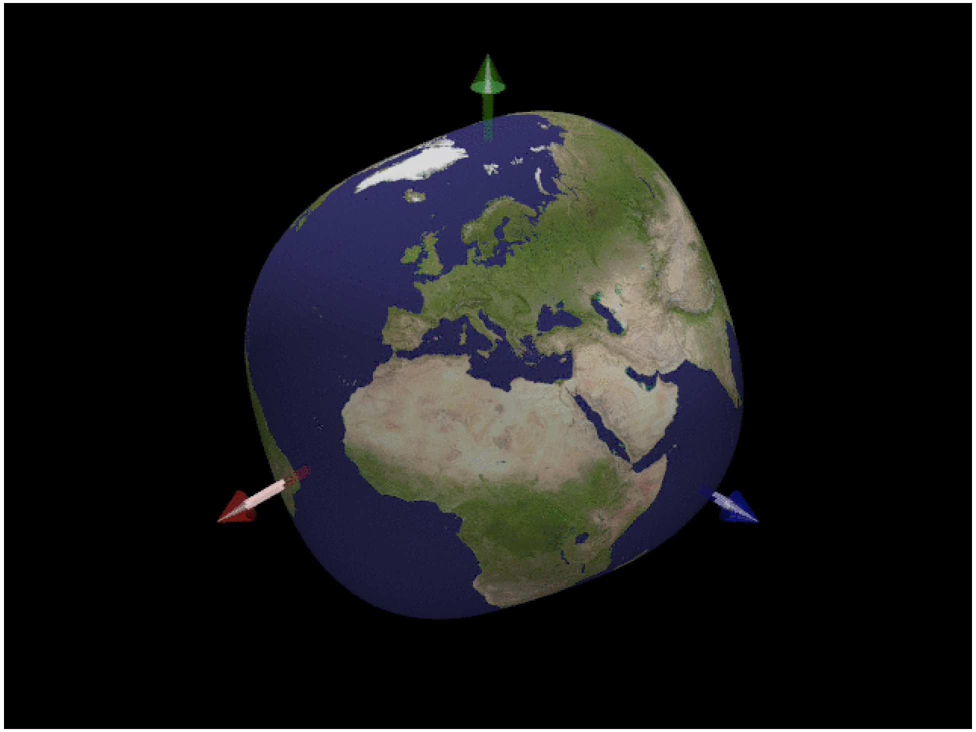
Which is your favorite bit of the Earth?



Normal modes care about velocity and density there!

Normal modes





Spheroidal, n=0, ℓ=4

period ≈ 26 minutes

Visualisations from http://icb.u-bourgogne.fr/nano/manapi/saviot/terre/index.en.html

Normal mode splitting

We have looked at the normal modes of a self gravitating earth. You will often hear the phrase SNREI Earth -

- Spherical – not elliptical
- Non-rotating – no days and nights
- Elastic – not attenuating
- Isotropic – not even relevant to PREM which is anisotropic in the upper mantle

These approximations are logical ones, but when we observe normal modes, we see that the degeneracy of the modes has been lost in many cases:

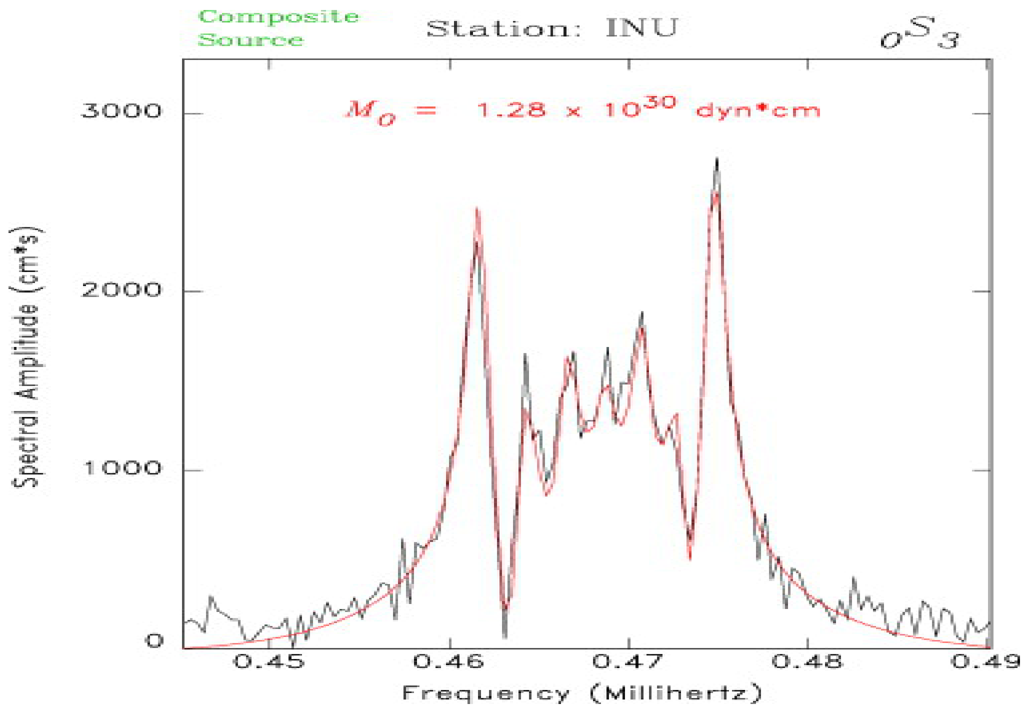
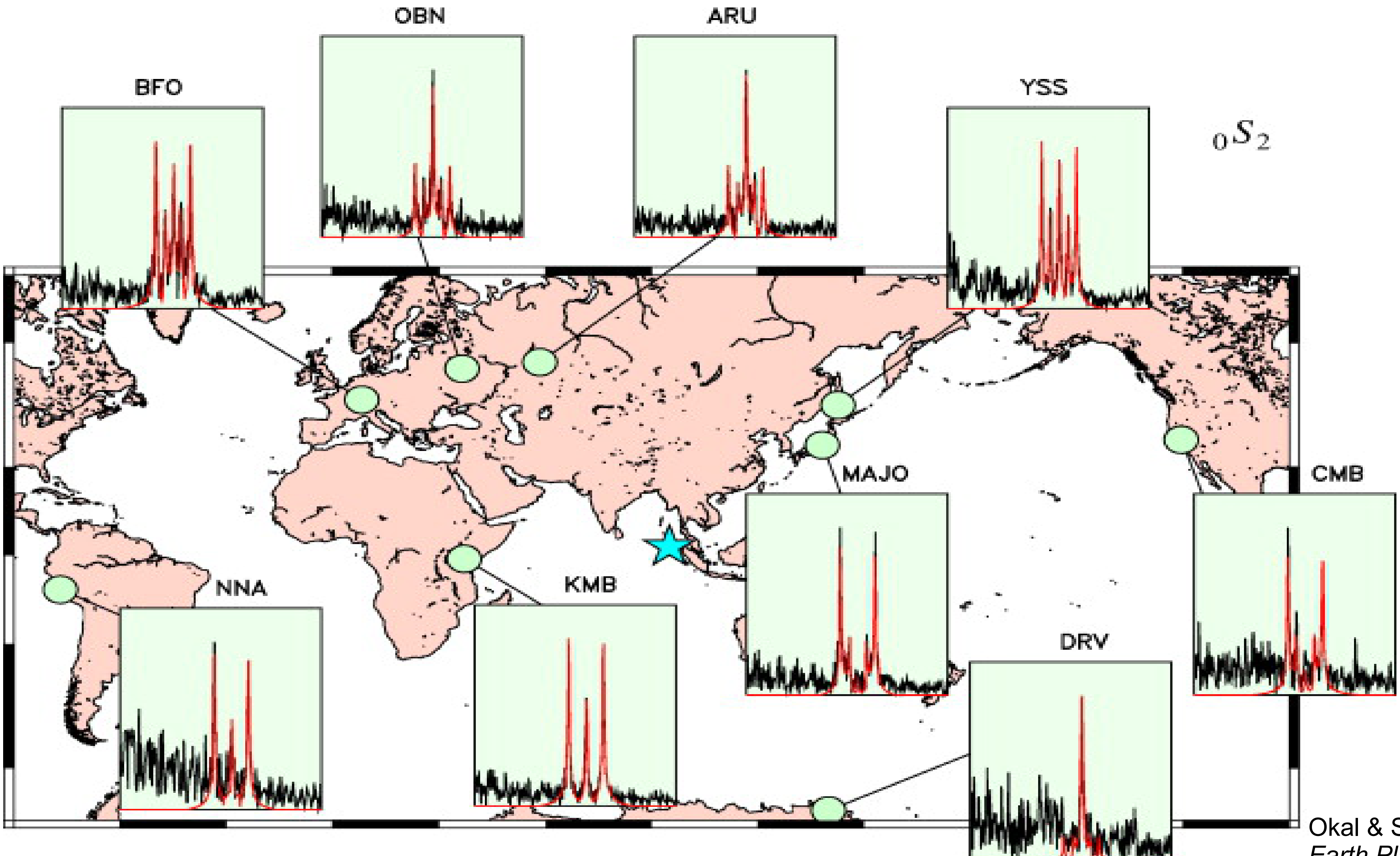


Figure 2. Spectrum of $_0S_3$ at Inuyama, Japan (INU). On this remarkable record, all seven singlets are resolvable (m = +2 only barely), and remarkably well fit using Tsai et al.'s (2005) composite source scaled to a full moment of 1.28 × 10^{30} dyn cm (synthetic in red).

Okal & Stein, *Phys. Earth Planet. Int.*, 2009

Normal mode splitting

This loss of degeneracy allows us to see the different singlets (different m values) which contribute to the multiplet in different locations. Again, we see mode 0S2.



Okal & Stein, *Phys. Earth Planet. Int.,* 2009

Splitting functions

The loss of degeneracy can be probed in terms of a 'splitting function'.

The terms of the splitting matrix, **H**, are given by:

$$H_{mm'} = \bar{\omega}_k \left[\underbrace{\left(a + mb + m^2 c\right)\delta_{mm'}}_{\text{Rotation \& ellipticity}} + \underbrace{\sum \gamma_s^{mm'} c_s^t}_{\text{3D elastic structure}} + i \underbrace{\sum \gamma_s^{mm'} d_s^t}_{\text{3D anelastic structure}} \right]$$

The anelastic 3D variations of the earth are poorly known, so we will not consider those here, though they are sometimes studied.

For this derivation, we follow Widmer-Schnidrig & Laske, *Treatise on Geophysics*, 2007, Chapter 1.03,

Splitting functions

We can then define structure coefficients c_s^t, often written as c_{st}, which contain the information about the 3D variation:

$$c_{st} = \int_0^{r_e} \boldsymbol{M}_s(r) \cdot \delta m_{st}(r) r^2 \, dr$$

where the **M** are kernels depending on things like U(r) and V(r), the eigenfunctions for a particular mode and K(r) (see Dahlen & Tromp appendix D4.2) which can be calculated for a known normal mode, and δ**m** are the 3D variations of the earth, which have been expressed in spherical harmonics so that δm_{st}, is the coefficient of the s,t spherical harmonic expansion.

We have integrated over the radius of the Earth, so that the structure coefficient tells us about the depth-average reaction of the singlet to the 3D variations in the Earth.

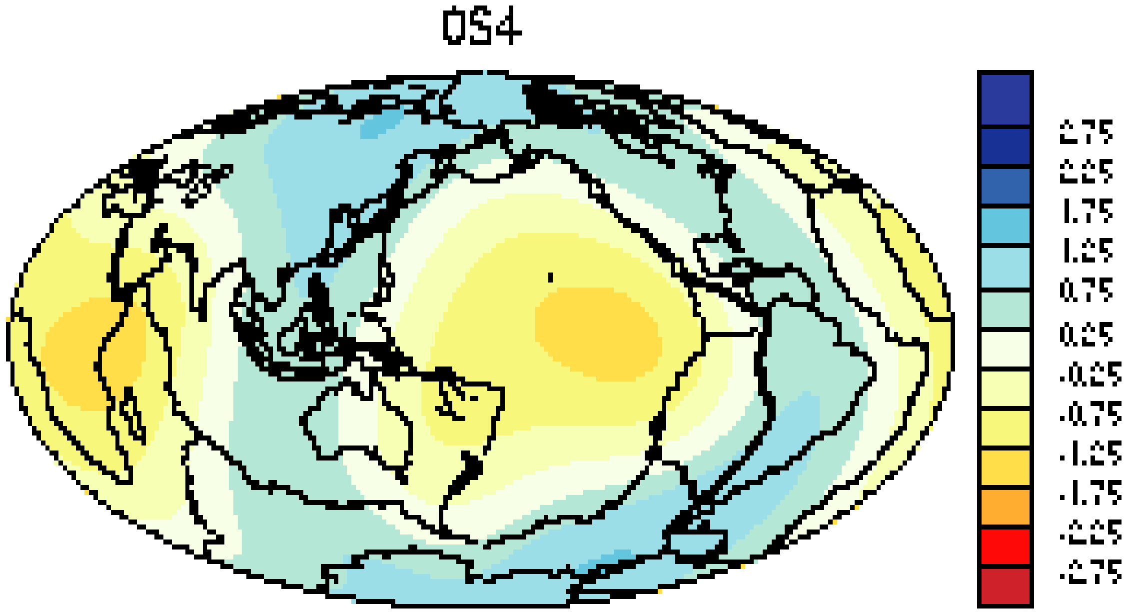
Finally, we can write down how all of the structure coefficients for a mode can be considered:

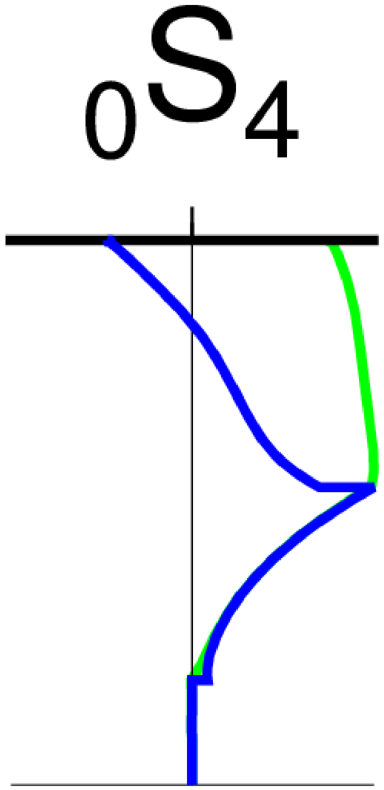
$$f_E(\theta, \phi) = \sum_{s,t} c_{st} Y_{st}(\theta, \phi)$$

For this derivation, we follow Widmer-Schnidrig & Laske, *Treatise on Geophysics*, 2007, Chapter 1.03,

Normal mode splitting

What do splitting coefficients look like?



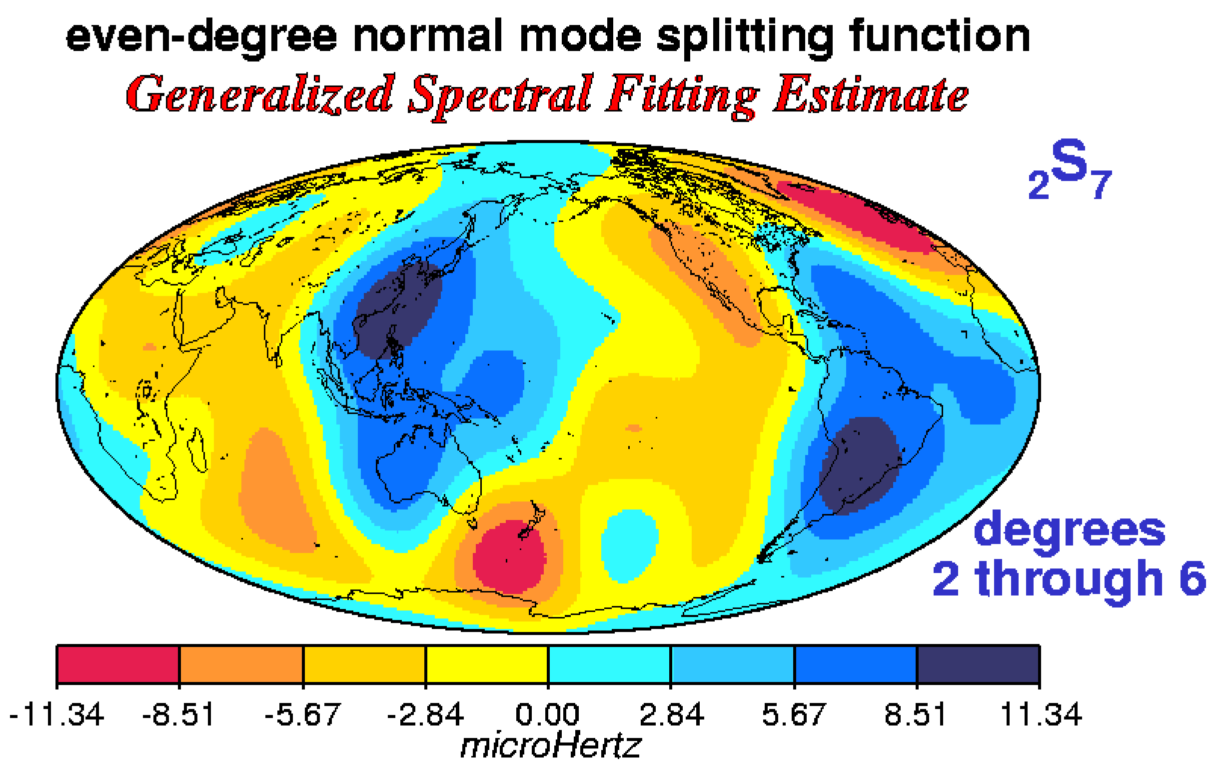


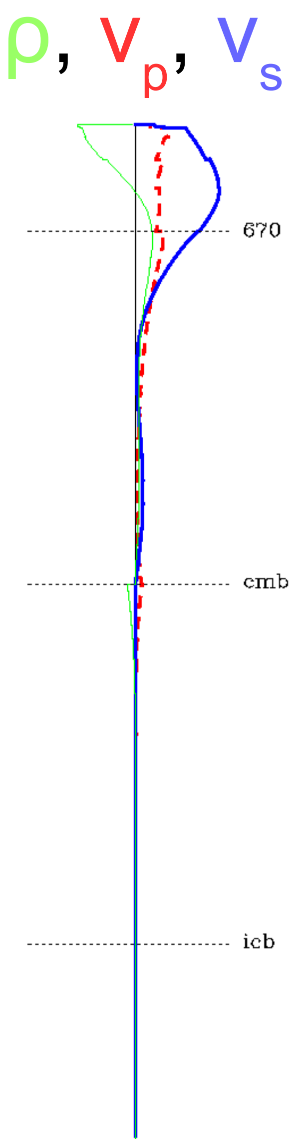
Splitting function from http://igppweb.ucsd.edu/~gabi/rem.dir/surface/splitting.sph.eps

Normal mode splitting

Splitting functions are a way of considering how a mode is affected by depth-averaged heterogeneities in Earth structure.

What do splitting coefficients look like?





Figures from Ritzwoller – see http://phys-geophys.colorado.edu/geophysics/nm.dir/nm.html

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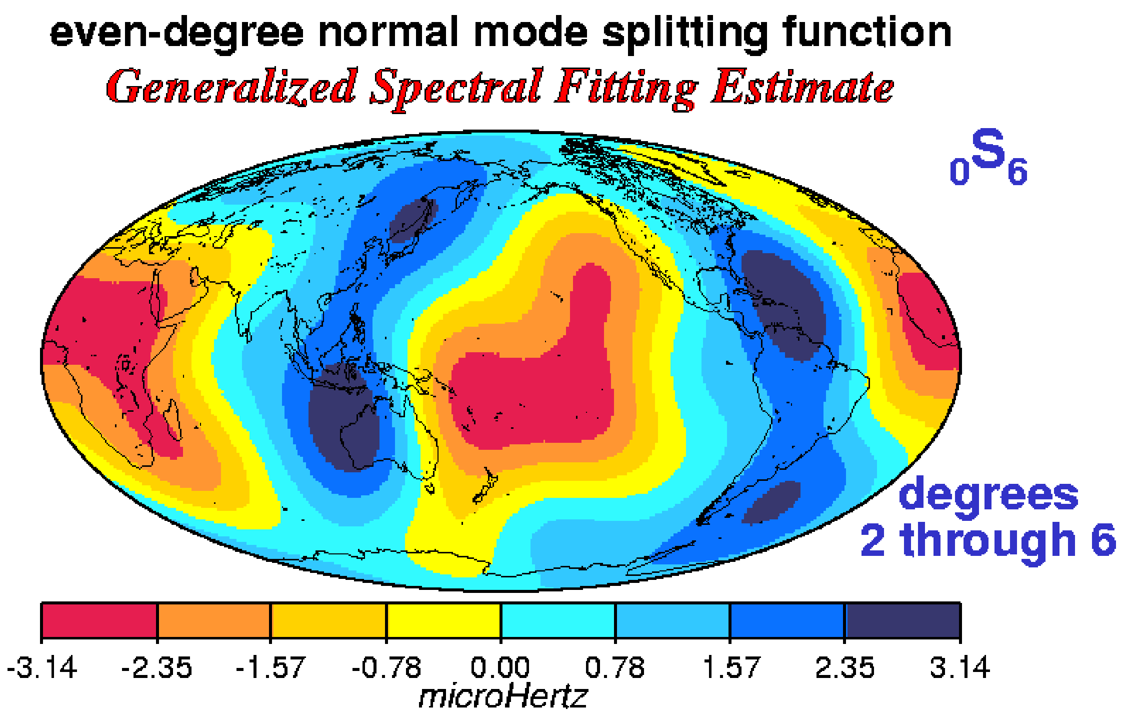
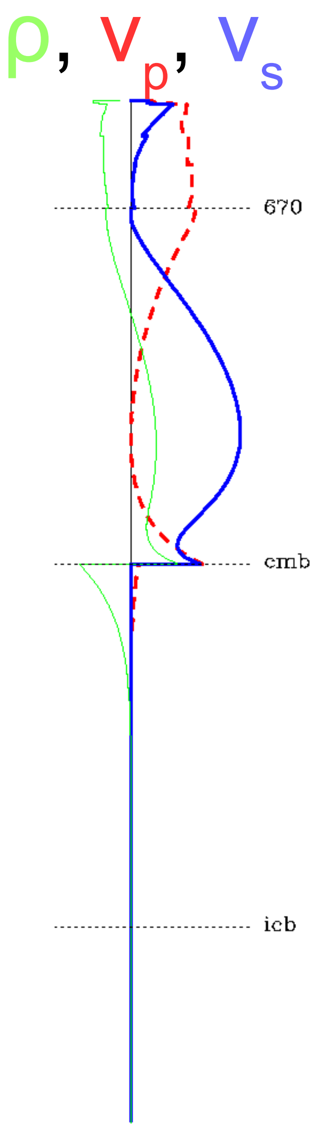




Figures from Ritzwoller – see http://phys-geophys.colorado.edu/geophysics/nm.dir/nm.html

A mode splitting caveat

What sorts of structure could splitting help us see?

There are symmetry and order requirements! For 3D <u>Earth</u> structure with angular and azimuthal orders s and t respectively to cause splitting of a <u>mode</u> with angular order l :

$$m-m'=t$$

i.e. axisymmetric (t=0) earth structure only affects the m=0 singlet.

$$l+s+l=0$$

i.e. s must be even to make a mode split.
<u>self coupled modes can only see even degree structure</u>.

$$0\leq s\leq 2l$$

i.e. modes cannot see structure which is too high in order

We follow Widmer-Schnidrig & Laske, *Treatise on Geophysics*, 2007, Chapter 1.03,

Normal mode coupling

At the lowest frequencies, coupling due to rotation is important. Ellipticity, 3D structure and attenuation also cause coupling:

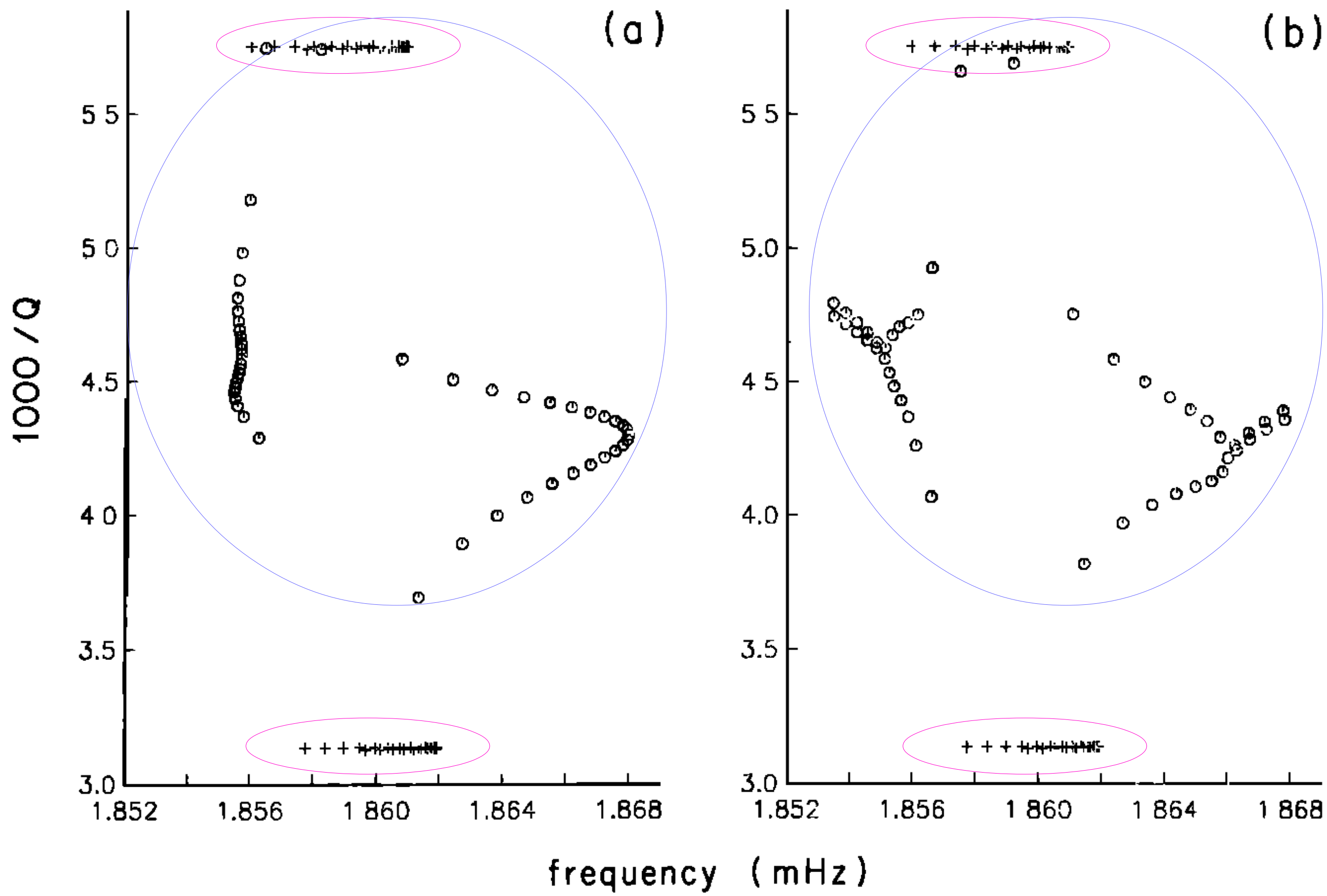


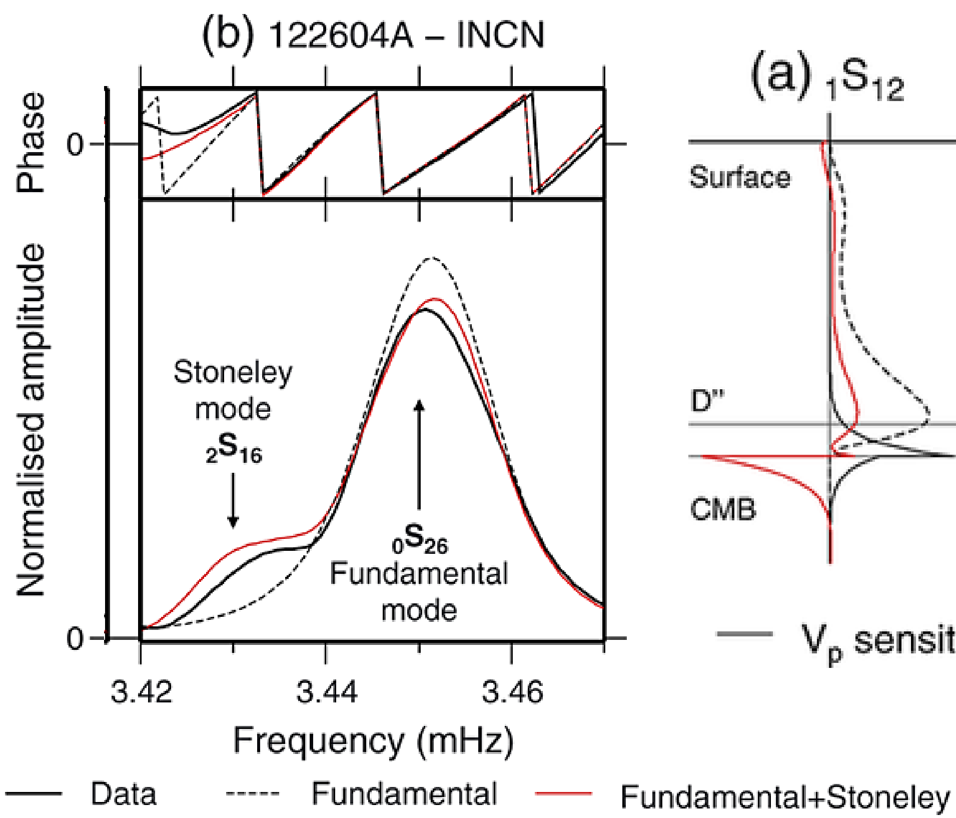
Fig. 4. Coupling between ${}_0S_{11}$ and ${}_0T_{12}$. Details as in Figure 2.

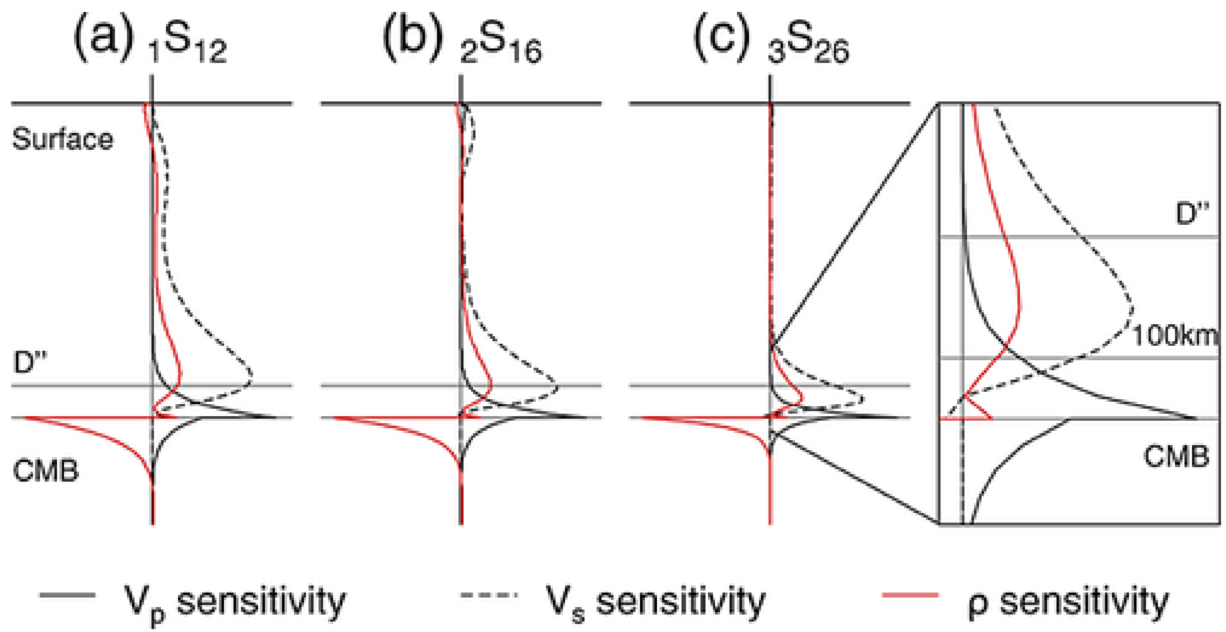
Fig. 2. Coupling between ${}_0S_{14}$ and ${}_0T_{15}$. (*a*) No coupling (plus signs) versus coupling (octagons) among all nearby (±1 mHz) fundamental modes due to rotation, ellipticity and attenuation. (*b*) No coupling (plus signs) versus coupling (octagons) between ${}_0S_{14}$ and ${}_0T_{15}$ due to rotation, ellipticity, attenuation, and aspherical structure of *Masters et al.* [1982].

From Masters et al, *J. Geophys. Res.,* 1983

Normal mode coupling

Mode coupling can let us measure the cross-coupled splitting functions of modes which are coupled together. This can let us see otherwise invisible/hard to see modes!





From Koelemeijer et al, *Geophys. Res. Lett,* 2013

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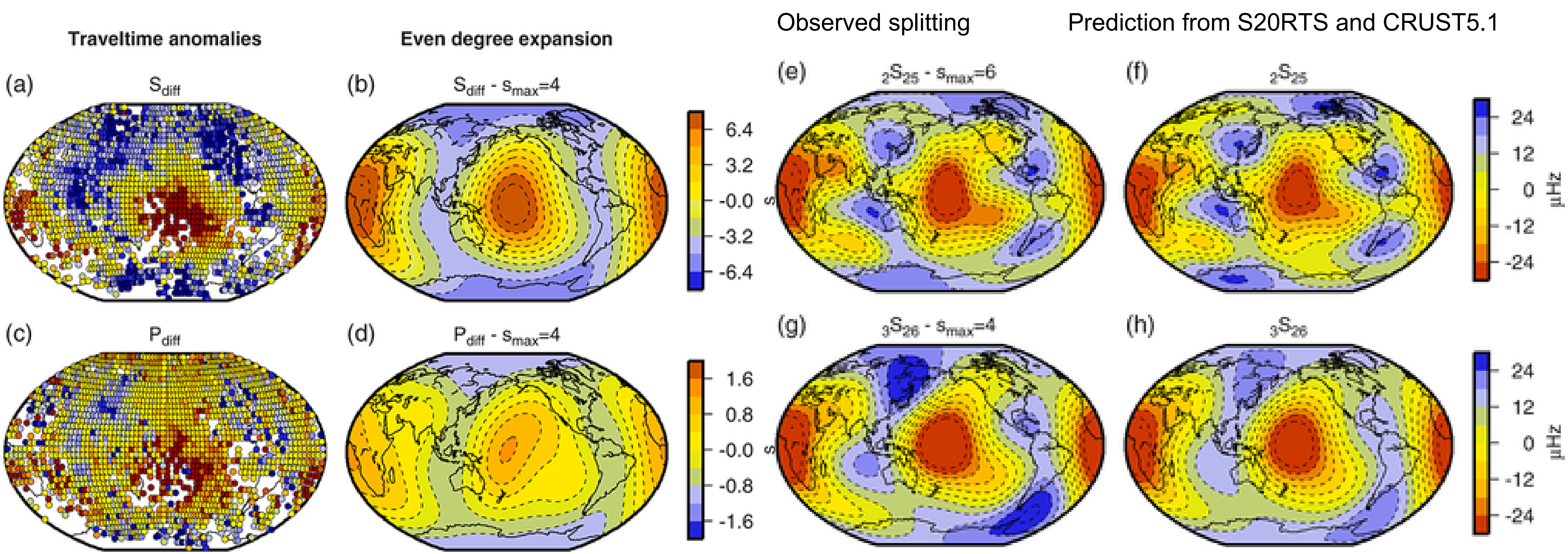

From Koelemeijer et al, *Geophys. Res. Lett,* 2013

A toroidal observation

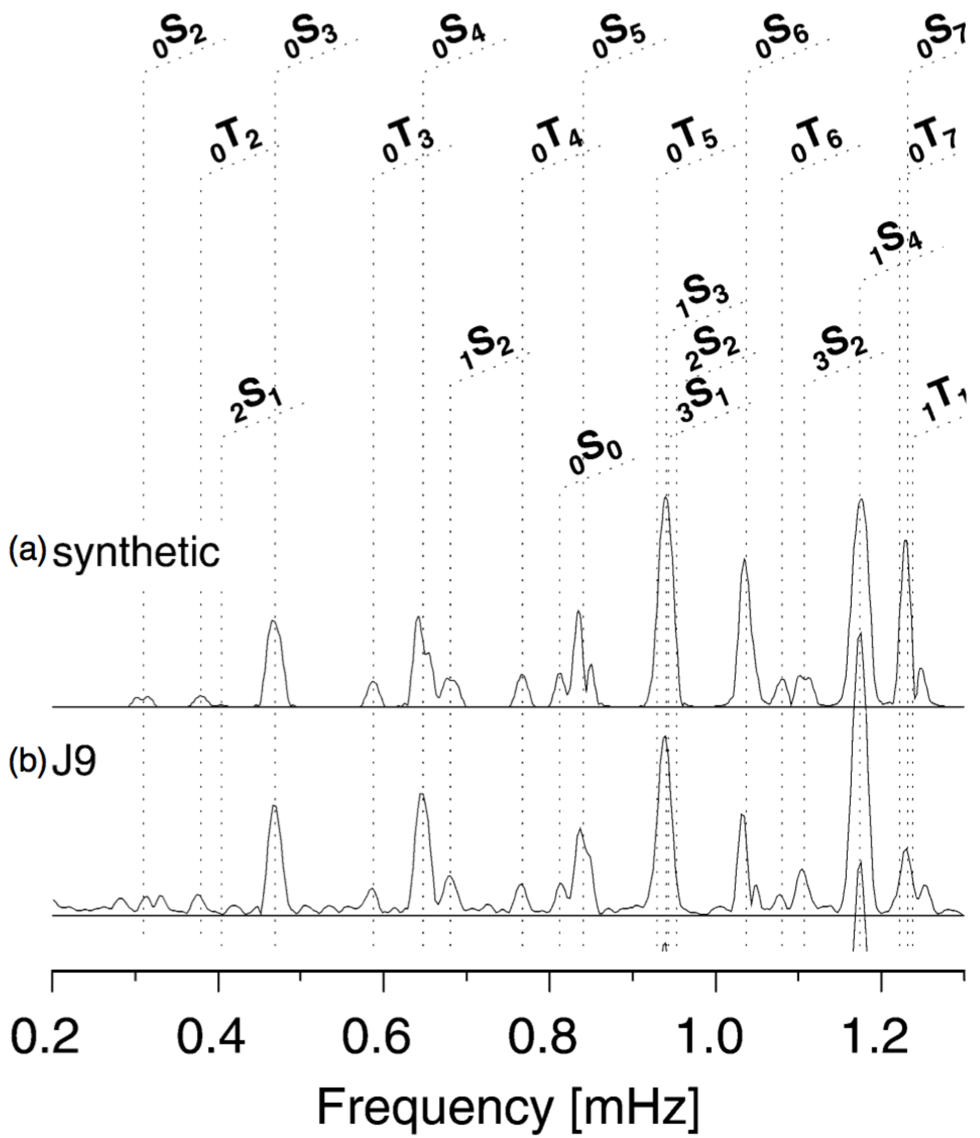

Figure 1. Comparison of amplitude spectra from the BIQ. The traces are (a) vertical-component coupled-mode synthetic for BFO including rotation, ellipticity of figure and heterogeneous elastic mantle structure, (b) C026 (J9, Strasbourg),

From Zurn et al, *Geophys. J.. Int.,* 2000