

# **Tutorials for Thermal Convection**

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## 1. Introduction

Thermal convection in planetary mantle is an important physical process that controls thermal evolution of terrestrial planets. Terrestrial planets contain radioactive heating in the mantle and primordial heating (i.e., resulted largely from the core formation). Thermal convection occurs in the mantle to release the heating and cool the planets. Tectonics and volcanisms are surface manifestations of this convective process.

Mantle convection may take different forms for different planets. On Earth, mantle convection involves recycling of the surface or oceanic lithosphere and results in plate tectonics. Because the lithosphere is relatively cold, recycling the lithosphere represents an extremely efficient way to release the heat and cool the mantle. On Venus and Mars, mantle convection appears to occur below a thick stagnant lid with rather different characteristics.

Mantle convection is a highly non-linear process with non-linear rheology and energy transfer. For this tutorial, we study basic characteristics of thermal convection by using a computer program called citcom (Moresi et al., 1996). Citcom solves the conservation equations of the mass, energy, and momentum for thermal convection problems by using a finite element method. For this system of equations, there are two controlling nondimensional parameters: Rayleigh number  $Ra$  and internal heating rate  $H$  ( $Ra$  is a measure of convective vigor). For a given  $Ra$  and  $H$ , citcom can be used to find heat flux at the surface and bottom boundary, flow velocity, temperature, and other important physical parameters.

## 2. Governing equations

The governing equations for thermal convection are the conservation equations of the mass, momentum, and energy. Over large time scales ( $>10,000$  years), the mantle may be approximated as incompressible fluid, and then the mass conservation becomes continuity equation. The large mantle viscosity or Prandtl number (viscosity divided by thermal diffusivity) enables us to ignore the inertial terms in the momentum equation. To the first order, we can also ignore viscous heating and adiabatic heating in the energy equation. The governing equations can be written as

$$\nabla \cdot \vec{u} = 0, \quad (1)$$

$$-\nabla P + \nabla \cdot [\eta(\nabla \vec{u} + \nabla^T \vec{u})] + \rho g \vec{e}_z = 0, \quad (2)$$

$$\rho c_p \left( \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T \right) = k \nabla^2 T + \rho H, \quad (3)$$

where  $u$ ,  $P$ ,  $\eta$ ,  $\rho$ , and  $T$  are the velocity, pressure, viscosity, density, and temperature, respectively;  $e_z$  is the unit vector in vertical direction;  $c_p$ ,  $k$ ,  $H$  and  $t$  are the specific heat, thermal conductivity, internal heating production rate and time, respectively.  $\rho$  can be related to coefficient of thermal expansion  $\alpha$  and reference surface density and temperature as

$$\rho = \rho_0 [1 - \alpha(T - T_0)]. \quad (4)$$

In fluid dynamics, we often like to deal with dimensionless numbers. For example, we normalize depth of mantle by the thickness of mantle, which yields dimensionless thickness of 1. The real advantage of this practice is to identify controlling parameters that result from nondimensionalize the governing equations. For example, for thermal convection problems, two important nondimensional parameters can be immediately identified, Rayleigh number,  $Ra$ , and internal heating parameter,  $H$ . Then for thermal convection problems, whether they are for the upper mantle convection, or whole mantle convection, or convection for Mars, as long as  $Ra$  and  $H$  are chosen to be the same, the dynamics as dictated by the nondimensional equations is the same.

Equations (1)-(3) are nondimensionalized using the following characteristic scales: length  $D$ ; time  $D^2/\kappa$ , viscosity  $\eta_0$ , temperature  $\Delta T$ , where  $D$ ,  $\kappa$ ,  $\eta_0$  and  $\Delta T$  are the thickness of the box, thermal diffusivity, reference viscosity, and temperature difference across the box depth. The non-dimensional equations are

$$\nabla \cdot \vec{u} = 0, \quad (5)$$

$$-\nabla P + \nabla \cdot [\eta(\nabla \vec{u} + \nabla^T \vec{u})] + RaT \vec{e}_z = 0, \quad (6)$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \nabla^2 T + Q, \quad (7)$$

where dimensionless parameters  $Ra$ , Rayleigh number, and  $Q$ , internal heating rate, are

$$Ra = \frac{\rho_0 g \alpha \Delta T D^3}{\eta_0 \kappa}, \quad (8)$$

$$Q = \frac{D^2 H}{c_p \kappa \Delta T}, \quad (9)$$

For this tutorial, we will assume  $Q$  to be zero, and such convection is often basal heating convection. This leaves only one controlling parameter:  $Ra$ .

### 3. Rheology and Constitutive Laws

The mantle rheology describes how rocks deform or flow under mantle  $T$  and  $P$  conditions. Two common ways to study the rheology are 1) to examine how rocks deform in the laboratory under conditions as realistic as one can achieve and 2) to model Earth's response to certain forces (e.g., post-glacial rebound). It is generally agreed that mantle rocks are much weaker and easier to deform under higher  $T$ . However, mantle viscosity increases with  $P$ . In some part of the mantle, deformation is accommodated via diffusion creep, while dislocation creep is the dominant deformation mechanism in other part of the mantle. For dislocation creep deformation, mantle viscosity also depends on stress and is controlled by what is often called power-law rheology.

For this tutorial, we will use the simplest form of rheology: constant viscosity throughout the box,  $\eta_0$ .

### 4. Boundary Conditions and Initial Conditions.

For any differential equations, we need to specify boundary conditions to have a unique solution. For time-dependent problems, we also need initial solutions. For our 2-D Cartesian models of this tutorial, we use free slip boundary conditions (i.e., zero normal velocity and zero shear stress) for the four sides of the box and isothermal boundary condition for the top and bottom boundaries (i.e.,  $T=0$  and  $1$  at the top and bottom respectively) and reflecting (or zero heat flux) boundary condition for the two sidewalls. However, other types of boundary conditions are often used as well, such as prescribed surface velocity or bottom heat flux.

Initial temperature is needed as an initial condition for the time-dependent energy equation (unlike wave equations which require both initial

displacement and velocity to specify the solution because of their 2nd order derivatives with time, our energy equation only has first order derivative with time and requires only one initial condition). In thermal convection studies, often we are interested in steady state solutions by running models for many time steps. Quite often, these steady state solutions (or statistically steady state) are rather insensitive to initial conditions.

For critical Ra and Nu-Ra calculations in this tutorial, the initial temperature is given as

$$T(x, z) = z + p \cos\left(\frac{\pi x}{L}\right) \sin(\pi z), \quad (10)$$

where p is the magnitude of the perturbation (e.g., several percent), and L is the nondimensional length of the box. For the tutorial problem of lithospheric cooling and subsequent instability, the initial condition is

$$T(x, z) = 1 + p[\text{rand}() - 0.5], \quad (11)$$

where rand() stands for a random number in the range of 0 and 1.

## 5. Numerical Methods and Grid

Citcom employs a multigrid finite element method that solves for velocity and pressure via a primitive variable formulations [e.g., Hughes, 1987]. It uses the stream-line upwind Petrov-Galekin (SUPG) for energy equation [e.g., Brooks, 1980]. Some of the implementation issues are discussed in Moresi and Solomatov [1995]. This version of code also includes some enhancements (full multigrid and consistent projection) that are discussed in Zhong et al. [2000]. Most of you as a user may not care that much about these issues, but if you do, you can always check up these references.

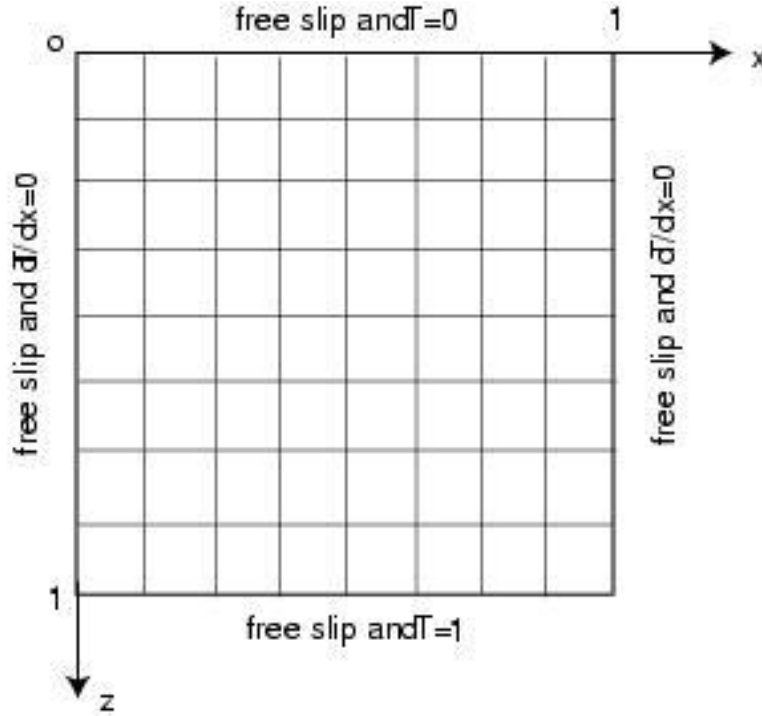


Figure 1. Coordinates and finite element grid. In this example, the number of elements in x and z directions,  $e_x$  and  $e_z$ , are both 8.

## 6. Tutorial topics

### 6.1. Calculations of critical Ra.

As we have learned, for a fluid layer with differential temperature  $\Delta T$  across the layer, when Rayleigh number is above a certain value,  $Ra_c$ , the fluid layer is unstable against perturbation and thermal convection will take place.  $Ra_c$  is the critical Rayleigh number. When Rayleigh number of the fluid layer is smaller than  $Ra_c$ , the layer is stable and thermal convection cannot occur. As shown in Turcotte and Schubert (2002), when initial temperature takes the form of equation (10) and the fluid layer has free-slip and isothermal boundary conditions,  $Ra_c$  depends on the wavelength of the perturbations (i.e.,  $L$  in equation (10)), and is given by

$$Ra_c = \frac{(\pi^2 + k^2)^3}{k^2}, \quad (12)$$

where  $k=2\pi/\lambda$  is the wavenumber, and  $\lambda$  is the nondimensional wavelength. For reflecting boundary conditions used here,  $\lambda=2L$  or  $k=\pi/L$ , where  $L$  is the nondimensional length of the box. Clearly, for a unit aspect of box with  $L=1$ ,  $Ra_c=8\pi^4=779.27$ .

For this part of the tutorial, we determine  $Ra_c$  using Citcom code and compare results with that from (12). We will follow the following strategy. As we know, for  $Ra < Ra_c$ , the initial perturbation given by (10) will decay away or horizontal gradient given in (11) will decrease with time. However, for  $Ra > Ra_c$ , the opposite will happen. A good measure is the kinetic energy of the flow for the box,  $K_E$ ,

$$K_E = 1/2 \iint \vec{u} \cdot \vec{u} dx dz . \quad (13)$$

$K_E$  increases/decreases as the perturbation increases/decreases. Therefore, for a given  $Ra$  in a calculation, by monitoring  $K_E$  with time, we can determine whether  $Ra$  is smaller or larger than  $Ra_c$ . We can run multiple calculations with different  $Ra$  to bracket or determine  $Ra_c$ .

Knowing  $Ra_c=779.27$  for a unit aspect box (i.e.,  $L=1$ ), you should first try to determine  $Ra_c$  using Citcom. Use input file `input_Racr` and try two cases with  $Ra=800$  and  $Ra=760$ , respectively, to see how  $KE$  varies with time for each case. In the output directory (e.g., `CASE1`), you will see a file `a.time_dependence` and the first and second columns of this file are time and  $K_E$ , respectively. You should try more cases to further narrow down numerically determined  $Ra_c$  to see how well you can determine  $Ra_c$ .

Here are some further questions to try or think about.

- A) Vary box length  $L$  to see how well you can reproduce results from (12).
- B) Does the magnitude of the perturbation  $p$  in (10) affect your results of  $Ra_c$ ?
- C) Equation (12) is obtained from the linear stability analysis that suggests that the velocities and perturbations grow exponentially with time for the initial stage. Can you check  $K_E$  from the output file to see how it varies with time?
- D) If the initial perturbation in equation (10) is modified as  $T(x, z) = z + p \cos(\frac{\pi x}{L}) \sin(2\pi z)$ , i.e., the  $z$  dependence is changed, do you think that  $Ra_c$  would change?

## 6.2. Nu-Ra scalings.

For entirely basal heating thermal convection with free-slip and isothermal boundary conditions, we have learned that as Ra increases, heat flux or Nusselt number Nu will also increase in a scaling form:  $Nu = aRa^{1/3}$ . We also know that thermal boundary layer thickness  $\delta$  should also scale as  $\delta \sim Ra^{-1/3}$  and flow velocity  $u \sim Ra^{2/3}$ .

Here we will compute multiple cases with different Ra in a unit aspect box and quantify outputs of heat flux Nu, flow velocity u and thermal boundary layer thickness  $\delta$  to see how well they follow the above scaling laws. Different from the first tutorial calculation that only needs the initial stage results, here calculations need to be computed to approximately steady state (i.e., until heat flux does not change appreciably with time) and depending on Ra, you may need to run the calculations for thousands of time steps. For practical reasons, do not use too high Ra (why?), and try  $Ra = 10^4$ ,  $3 \times 10^4$ ,  $10^5$ ,  $3 \times 10^5$ ,  $10^6$ , and  $3 \times 10^6$ .

Use input file `input_Nu_Ra` to do calculations with different Ra. To check the time dependent heat flux and characteristic flow velocity, go to the file `a.time_dependence`, the 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup> and 6<sup>th</sup> columns are for surface heat flux (Nu), bottom heat flux, surface averaged velocity and bottom averaged velocity, respectively (the first column is the time). Check to see whether they are in the steady state. Only use the steady state results of surface Nu and velocities for fitting scaling laws for different Ra. You may need to use software package `grace` or `matlab` to fit the scaling laws for a set of Nu-Ra values or u-Ra values.

You should also use GMT script `plot_temperature_velo` to plot temperature and flow velocities and also surface and bottom heat fluxes at different time steps. Pay attention to flow and temperature patterns (e.g., hot upwellings) and also how surface and bottom heat flux patterns are related to the interior temperature and flow patterns. You will need to modify `plot_temperature_velo` to suit your need.

Here are some further questions to try or think about.

- A) How would you define thermal boundary layer and determine its thickness  $\delta$ ? Hint that there are output files `a.ave_z.timesteps` for



horizontally averaged temperature (the second column) versus depth  $z$  (the first column).

- B) If you can determine  $\delta$ , can you try to fit the scaling  $\delta \sim Ra$  to see whether it is what you expect?
- C) What do you need to do for your numerical grids when  $Ra$  is large? Why?

### 6.3. Cooling and thickening of lithosphere and subsequent instability.

For a box with insulating sidewalls and uniform temperature  $T=1$  everywhere except for the surface where  $T=0$  is subscribed as boundary conditions (the bottom boundary is kept at  $T=1$ ), the box will cool from the above, and a top thermal boundary layer will develop and thicken with time. As long as heat conduction is the dominant mode of heat transfer and the thermal boundary layer is much thinner than the thickness of the box, the time evolution of the temperature field  $T(z,t)$  is governed by the well-known cooling half-space model (e.g., Turcotte and Schubert, 2002). However, the top thermal boundary layer is gravitationally unstable. One can understand the process as following. When the top boundary layer thickness  $\delta$  reaches a certain value, such that  $Ra_\delta = \rho g \alpha \Delta T \delta^3 / (\eta \kappa)$  reaches a critical value  $Ra_{cd}$ , then the thermal boundary layer becomes unstable, and the boundary layer destabilizes and drives convection.

We may derive the time it takes for the instability to occur, or the onset time,  $\tau$ . The top boundary layer thickness  $\delta$  thickens with time  $t$ , as the following:

$$\delta \sim (\kappa t)^{1/2}, \quad (14)$$

At the onset time  $\tau$ ,

$$Ra_\delta = C' \rho g \alpha \Delta T \kappa^{1/2} \tau^{3/2} / (\eta) = Ra_{cd}, \quad (15)$$

where  $C'$  is a constant from (14). Assuming that  $Ra_{cd}$  as a critical  $Ra$  is a constant, (15) becomes

$$\tau^{3/2} = \eta / (C \rho g \alpha \Delta T \kappa^{1/2}) = C (D^3 / \kappa^{3/2}) [\eta \kappa / (\rho g \alpha \Delta T D^3)]. \quad (16)$$

$$\tau = a (D^2 / \kappa) [\eta \kappa / (\rho g \alpha \Delta T D^3)]^{2/3} = a (D^2 / \kappa) Ra^{-2/3}, \quad (17)$$

where  $a$  is a constant and  $Ra$  is the Rayleigh number defined by layer thickness  $D$ . Since the time is normalized by  $D^2/\kappa$ , dimensionless onset time

$$\tau = aRa^{-2/3}. \quad (18)$$

The goal of this tutorial is to test the cooling half-space model for the initial heat conduction process and then to verify the scaling relation for the onset time (18). According to the cooling half-space model (Turcotte and Schubert, 2002),

$$\frac{T - T_s}{T_m - T_s} = \text{erf} \left[ \frac{z}{2(\kappa t)^{1/2}} \right], \quad (19)$$

where  $T_s$  and  $T_m$  are the surface and interior temperatures, respectively,  $z$  is the depth, and  $\text{erf}()$  stands for error function. The dimensionless form of (19), using the same scalings as for our convection problem, is

$$T = \text{erf} \left[ \frac{z}{2(t)^{1/2}} \right], \quad (20)$$

which can be directly compared with results from Citcom.

Use input file `input_lith` for the model calculations where  $Ra$  can be varied (note that a model with different  $Ra$  can be viewed as that with different fluid viscosity). For this type of calculations, equation (10) is used as the initial temperature condition that includes some random perturbations. Output file `a.ave_z.timestep` gives horizontally averaged temperature  $T_{\text{ave}}(z)$  (the second column) versus depth  $z$  (the first column) at some given time steps.  $T_{\text{ave}}(z)$  from Citcom can be compared with analytical solutions at the corresponding time from the cooling half-space model (i.e., equation 20) that is given in the third column in file `a.ave_z.timestep`. When the instability starts,  $T_{\text{ave}}(z)$  from Citcom will start to deviate from equation (20). Onset time  $t$  is defined as the time at which  $T_{\text{ave}}(z)$  from Citcom deviates from the analytical solutions by 1% at any depth (e.g., Huang et al., 2003).

To verify the onset time scaling (18), you will need to run several calculations with different  $Ra$ . For each calculation, determine onset time  $\tau$ . Then use `gnuplot` software package or `matlab` or any other method to fit a scaling for a set of  $\tau$ - $Ra$  values. Also use GMT script `plot_temperature_velo`

to make some plots at different time steps and regimes (e.g., conduction and instabilities).

Here are some further questions to try or think about.

- A) We define  $Ra_\delta = \rho g \alpha \Delta T \delta^3 / (\eta \kappa)$  using thermal boundary layer thickness. You may find the dimensionless thickness of the top thermal boundary layer  $\delta$  at the onset of instability from this exercise. Then we have  $Ra_\delta = \delta^3 Ra$ , where  $Ra$  is Rayleigh number defined by the box thickness and is given in input file. How does  $Ra_\delta$  compare between different model runs with different  $Ra$  and compare with critical  $Ra$  qualitatively from 6.1?
- B) You may find an expression for surface heat flux for the cooling half-space model from Turcotte and Schubert (2002). Citcom outputs surface heat flux at different time, as discussed before. Do you want to check how Citcom solutions compare with the analytical solutions? You may need to non-dimensionalize surface heat flux for comparison.
- C) After the onset of instability, what happens to the temperatures of the fluid interior and the top boundary layer, compared with the cooling half-space model prediction?
- D) This model and scenario can be compared with the evolution of oceanic lithosphere from the spreading centers (where its age is zero) to old ocean basins. Considering that our current model only uses constant viscosity and ignores temperature-dependent viscosity, what would you say how this idea may be applied to the oceanic lithosphere?

## 7. Use the code, input and output files

### Use the code

For all the tutorial topics discussed here, we will use a single executable `citcom.x`. In your work directory where the source code is contained, type unix/linux command

```
$make all
```

and `citcom.x` (and other post-processing executable codes) will be generated and be ready to use.

To run Citcom calculation, issue the following command

```
$citcom.x input_Racr
```

where input\_Racr is the input file that is specific to the run. The run may take a few minutes or hours, depending on the number of time steps and number of finite element nodes. The run will produce outputs in a directory specified in the input file (e.g., CASE1 under the work directory).

### Input files

We now discuss input files. There are three types of input files: input\_Racr, input\_Nu\_Ra, and input\_lith, that are used for tutorial topics 6.1, 6.2 and 6.3. First, let us look at input\_Racr, input file for computing critical Ra, and we will only discuss relevant parameters (i.e., those that you may want to vary).

```
Datafile="CASE1/a"
```

This specifies where output files are to be stored (e.g., directory CASE1). "a" here is the prefix for all the output files for this calculation. Notice that you must create directory CASE1 first (e.g., using command: \$mkdir CASE1) before running the case.

```
rayleigh=8.e2
```

This specifies Rayleigh number for this calculation. For another calculation, you may change it to some other value (e.g., 7.6e2, and prefix for output files be "b").

```
perturbmag=0.05
```

This is the magnitude of the perturbations  $p$  in equation (10).

```
maxstep=100
```

```
storage_spacing=2
```

They specify the total number of time steps and how frequently the outputs (e.g., kinetic energy) are generated (e.g., every 2 time steps). For the critical

Ra calculation, there is no need to compute too many time steps, but more frequent outputs are useful.

```
dimenx=1.0  
dimenz=1.0
```

They represent dimensionless length and thickness of the box (i.e., unit aspect ratio). If you would like to compute critical Ra for wavelength  $\lambda=1$ , you may want to reduce `dimenx=0.5`. Notice that there is a parameter “`perturb=1.0`” in the input file, and keep it to be 1.0.

```
mgunitx=2  
mgunitz=2  
levels=4
```

They specify the number of elements in each of x and z directions and multi-grid levels, respectively. The number of element in x or z direction is given by the equation:  $elx = mgunitx * 2^{(levels-1)}$ . In this case,  $elx=elz=16$ , and the number of nodes in each direction  $nox=noz=17$ .

In most cases, you do not need to change other parameters in the input file.

Now let’s look at input file for Nu-Ra calculations, `input_Nu_Ra`. This input file has nearly identical form as `input_Racr`. Parameter “`rayleigh`” should be larger than  $3e3$ , as discussed in section 6.2. Keep parameter “`dimenx`” to be 1, but you are welcome to increase it to some larger values (i.e., longer boxes). For this tutorial topic, parameter “`maxstep`” needs to be 1000s or more to get steady state solutions of heat flux. You may also need to use more numerical resolutions (i.e., more grid points), depending on Ra. Notice that parameter “`storage_spacing`” is set to be 20 here. This number can be even larger, given that the calculations use 1000s time steps and we are interested only in steady state solutions.

The third input file, `input_lith`, has a lot of more similarity to `input_Racr`, in the sense that we are interested in the transient process or results for the first 100’s time steps, and a relatively small “`storage_spacing`” helps to determine the onset time more precisely. However, if you would like to see the convective structure after the onset of instability, you may need to run for more time steps than did for critical Ra calculations. Note that there is a small switch in input files to determine which initial conditions (equations

10 or 11) are to be used for these three cases. However, you do not need to worry about it, as long as you use the input files correctly for the right cases.

### Output files

We now discuss output files from citcom runs. First, the output files are stored in directory with prefix that are specified in input file, as discussed before. For example, “CASE1/a” in the input file means that the files are stored in directory CASE1 with prefix “a”. You must create directory CASE1 before starting the calculation.

In the directory, you will see typically output files like: a.velo.XXXX, a.heatflux\_tb.XXXX, a.ave\_z.XXXX, and a.time\_dependence, where XXXX is time step. File a.velo.XXXX contains coordinates, temperature, velocities for each grid node at time step XXXX. Files a.heatflux\_tb.XXXX are for surface and bottom heat flux versus x coordinates for the given time step, while a.ave\_z.XXXX is for horizontally averaged temperature. File a.time\_dependence is for kinetic energy, averaged surface and bottom heat fluxes at different times.

Of course, it is not easy to see and understand the results by looking at the data files, and graphics are big helps here. GMT script plot\_temperature\_velo processes citcom output files and makes plots of thermal structure and velocity field in the box, and also surface and bottom heat flux. You may need to plot time-dependent kinetic energy or surface heat flux in file a.time\_dependence. You can try out your own GMT script for this purpose.

## 8. References

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